Lecture 12. LINES ON HYPERSURFACES

In this lecture we shall give an application of the theory of dimension. Consider the following problem. Let $X = V(F)$ be a projective hypersurface of degree $d = \deg F$ in $\mathbf{P}^n(K)$. Does it contain a linear subspace of given dimension, and if it does, how many? Consider the simplest case when $d = 2$ (the case $d = 1$ is obviously trivial). Then F is a quadratic form in $n+1$ variables. Let us assume for simplicity that $\mathrm{char}(K) \neq 2$. Then a linear m-dimensional subspace of dimension in $V(F)$ corresponds to a vector subspace L of dimension $m+1$ in K^{n+1} contained in the set of zeroes of F in K^{n+1}. This is an isotropic subspace of the quadratic form F. From the theory of quadratic forms we know that each isotropic subspace is contained in a maximal isotropic subspace of dimension $n+1-r+[r/2]$, where r is the rank of F. Thus $V(F)$ contains linear subspaces of dimension $\leq n-r+[r/2]$ but does not contain linear subspaces of larger dimension. For example, if $n = 3$, and $r = 4$, F is isomorphic to $V(G)$, where G is given by the equations

$$T_0T_1 - T_2T_3 = 0.$$

For every $\lambda, \mu \in K$, we have a line $L(\lambda, \mu)$ given by the equations

$$\lambda T_0 + \mu T_2 = 0, \mu T_1 + \lambda T_3 = 0,$$

or a line $M(\lambda, \mu)$ given by the equation

$$M(\lambda, \mu) : \lambda T_0 + \mu T_3 = 0, \mu T_1 + \lambda T_2 = 0.$$

It is clear that $L(\lambda, \mu) \cap L(\lambda', \mu') = \emptyset$ (resp. $M(\lambda, \mu) \cap M(\lambda', \mu') \neq \emptyset$) if and only if $(\lambda, \mu) \neq (\lambda', \mu')$ as points in $\mathbf{P}^1(K)$. On the hand $L(\lambda, \mu) \cap M(\lambda', \mu')$ is one point always. Under an isomorphism $V(F) \cong \mathbf{P}^1(K) \times \mathbf{P}^1(K)$, the two families of lines $L(\lambda, \mu)$ and $M(\lambda, \mu)$ correspond to the fibres of the two projections $\mathbf{P}^1(K) \times \mathbf{P}^1(K) \to \mathbf{P}^1(K)$.

Another example is the *Fermat hypersurface* of $V(F) \subset \mathbf{P}^3(K)$ of degree d, where

$$F = T_0^d + T_1^d + T_2^d + T_3^d.$$

Since

$$T_i^d + T_j^d = \prod_{s=1}^{d} (T_i + \rho^s T_j))$$

where ρ is a primitive d-th root of -1, we see that $V(F)$ contains $3d^2$ lines. Each one is defined by the equations of the type:

$$T_i + \rho^s T_j = 0, T_k + \rho^t T_l = 0,$$

where $\{i,j,k,l\} = \{0,1,2,3\}$. In particular, when $d = 3$, we obtain 27 lines. As we shall see in this Lecture, "almost every" cubic surface contains exactly 27 lines. On the other hand if $d \geq 4$, "almost no" surface contains a line.

To solve our problems, we first parametrize the set of linear r-dimensional subspaces of of $\mathbf{P}^n(K)$ by some projective algebraic set. This is based on the classic construction of the Grassmann variety.

Let V be a vector space of dimension $n+1$ over a field K and let L be its linear subspace of dimension $r+1$. Then the exterior product $\bigwedge^{r+1}(L)$ can be identified with a one-dimensional subspace of $\bigwedge^{r+1}(V)$, i.e., with a point $[L]$ of the projective space $\mathbf{P}(\bigwedge^{r+1}(V)) = \bigwedge^{r+1}(V)\setminus\{0\}/K^*$. In coordinates, if $e_1,\ldots,e_{n+1}$ is a basis of V, and $f_1,\ldots,f_{r+1}$ is a basis of L, then $\bigwedge^{r+1}(L)$ is spanned by one vector

$$f_1 \wedge \ldots \wedge f_{r+1} = \sum_{1\leq i_1<\ldots<i_{r+1}\leq n+1} p[i_1,\ldots,i_{r+1}]e_{i_1} \wedge \ldots e_{i_{r+1}}.$$

If we order the vectors $e_{i_1} \wedge \ldots e_{i_{r+1}}$ we may identify $\bigwedge^{r+1}(V)$ with $K^{\binom{n+1}{r+1}}$, then the coordinate vector of the point $[L]$ in $\mathbf{P}(\bigwedge^{r+1}(V)) \cong \mathbf{P}^{\binom{n+1}{r+1}-1}(K)$ is the vector $(\ldots,p[i_1,\ldots,i_{r+1}],\ldots)$. The coordinates $p[i_1,\ldots,i_{r+1}]$ are called the *Plücker coordinates* of L. If we denote by $M(L)$ the matrix of size $(r+1)\times(n+1)$ with the j-th row formed by the coordinates of f_j with respect to the basis $(e_0,\ldots,e_{n+1})$, then $p[i_1,\ldots,i_{r+1}]$ is equal to the maximal size minor of $M(L)$ composed of the columns $A_{i_1},\ldots,A_{i_{r+1}}$.

The next theorem shows that the correspondence $L \to [L]$ is a bijective map from the set of linear subspaces of dimension $r+1$ in V to the set of K-points of a certain projective subset $G(r+1,n+1)$ in $\mathbf{P}^{\binom{n+1}{r+1}-1}(K)$.

Theorem 1. *The subset $G(r+1,n+1)$ of lines in $\bigwedge^{r+1}(V)$ spanned by decomposable $(r+1)$-vectors $f_1 \wedge \ldots \wedge f_{r+1}$ is a projective algebraic set in $\mathbf{P}(\bigwedge^{r+1}(V)) \cong \mathbf{P}^{\binom{n+1}{r+1}-1}(K)$. The map $L \to [L] = \bigwedge^{r+1}(L)$ is a bijective map from the set of linear subspaces of V of dimension $r+1$ to the set $G(r+1,n+1)$.*

Proof. We use the following fact from linear algebra. For every $t \in \bigwedge^{r+1}(V)$ let $L(t) = \{x \in V : t\wedge x = 0\}$. This is a linear subspace of V. Then $\dim L(t) \geq r+1$ if and only if t is decomposable and equal to $f_1 \wedge \ldots \wedge f_{r+1}$ for some linear independent vectors $f_1,\ldots,f_{r+1}$ which have to form a basis of $L(t)$. This assertion shows that the subspace L can be reconstructed uniquely from $[L]$ as the subspace $L(t)$, where t is any basis of $[L]$. Let us prove the assertion. The sufficiency is easy. If $t = f_1 \wedge \ldots \wedge f_{r+1}$ for some basis $\{f_1,\ldots,f_{r+1}\}$ of a linear subspace of dimension $r+1$, then, obviously, $f_1 \wedge \ldots \wedge f_{r+1} \wedge x = 0$ for any $x \in L = Kf_1 + \ldots + Kf_{r+1}$ so that $L \subset L(t)$. Since $f_1 \wedge \ldots \wedge f_{r+1} \wedge x = 0$ implies that $\bigwedge^{r+2}(Kf_1 + \ldots + Kf_{r+1} + Kx) = 0$, we obtain that $\dim(Kf_1 + \ldots + Kf_{r+1} + Kx) = r+1$, hence $x \in Kf_1 + \ldots + f_{r+1}$. This shows that $L = L(t)$. Conversely assume $\dim L(t) = r+1$. Let $f_1,\ldots f_{r+1}$ be a set of linear independent vectors in $L(t)$, and let $\{f_1,\ldots,f_{r+1},f_{r+1},\ldots,f_{n+1}\}$ be an extension of $\{f_1,\ldots,f_{r+1}\}$ to a basis of V. We can write

$$t = \sum_{i_1<\ldots<i_{r+1}} a_{i_1\ldots i_{r+1}} f_{i_1} \wedge \ldots \wedge f_{i_{r+1}}.$$

It is easy to see that $t\wedge f_i = 0, i = 1,\ldots,r+1$, implies $a_{i_1\ldots i_{r+1}} = 0$ for $\{i_1,\ldots i_{r+1}\} \neq \{1,\ldots,r+1\}$. Hence t is proportional to $f_1 \wedge \ldots \wedge f_{r+1}$.

To see why decomposable non-zero $(r+1)$-vectors define a closed subset $G(r+1,n+1)$ of $\mathbf{P}(\bigwedge^{r+1}(V))$ it suffices to observe that $\dim L(t) \geq r+1$ if and only if $rk(T_t) \leq n-r$, where T_t is the linear map $V \to \bigwedge^{r+2}(V)$ defined by the formula $x \mapsto t \wedge x$. The latter condition is equivalent to vanishing of $(n-r+1)$-minors of the matrix of T_t with respect to some basis. By taking a basis $e_1, \ldots, e_{n+1}$ of V, it is easy to see that the entries of the matrix of T_t are the Plücker coordinates of the space $L(t)$. Thus we obtain that $G(r+1,n+1)$ is the set of zeroes of a set of homogeneous polynomials of degree $n-r+1$. Observe that these polynomials have integer coefficients, so $G(r+1,n+1)$ is a projective k-set for any $k \subset K$.

More generally, let us fix a field k and consider the functor $G_k(r+1,n+1)$ from the category of k-algebras to the category of sets defined by:

$$G_k(r+1,n+1)(K) = \{\text{direct summands of } K^{n+1} \text{ of rank } r+1\}.$$

Note that a direct summand of a free module is a projective module. It is clear that the restriction of this functor to the subcategory of field extensions of k coincides with the functor

$$K \to \{\text{linear subspaces of } K^{n+1} \text{ of dimension } r+1\}.$$

The operation of exterior power, $M \to \bigwedge^{r+1}(M)$ defines the morphism of functors

$$p : G_k(r+1,n+1) \to \mathbf{P}_k^{\binom{n+1}{r+1}-1}.$$

If $r=0$ this morphism is an isomorphism. We leave to the reader to prove the following:

Theorem 2. *The morphism of functors $p : G_k(r+1,n+1) \to \mathbf{P}_k^{\binom{n+1}{r+1}-1}$ is an isomorphism onto a projective subvariety of $\mathbf{P}_k^{\binom{n+1}{r+1}-1}$.*

Definition. The projective variety $G_k(r+1,n+1)$ is called the *Grassmann variety* over the field k. The morphism p is called the *Plücker embedding* of $G_k(r+1,n+1)$. For every algebraically closed field K containing k, we shall identify the set $G_k(r+1,n+1)(K)$ with the projective algebraic subset $G(r+1,n+1)$ of $\mathbf{P}_k^{\binom{n+1}{r+1}-1}$.

Proposition 1. *The projective algebraic set $G(r+1,n+1)$ is an irreducible projective set of dimension $(n-r)(r+1)$.*

Proof. We shall give two different proofs of this result. Each one carries some additional information about $G(r+1,n+1)$. In the first one we use the following obvious fact: the general linear group $GL(n+1,K)$ acts transitively on the set of $(r+1)$-dimensional linear subspaces of K^{n+1}. Moreover the stabilizer of each such subspace L is isomorphic to the subgroup P of $GL(n+1,K)$ that consists of matrices of the form:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A, B, C are matrices of size $(r+1)\times(r+1)$, $(r+1)\times(n-r)$, $(n-r)\times(n-r)$, respectively. Let us consider $GL(n+1,K)$ as a closed subset of K^{n^2+1} defined by the equation $T_0 det((T_{ij})) - 1 = 0$. then it is clear that P is a closed subset of $GL(n+1,K)$ defined by the additional equations $T_{ij} = 0, i = n+2-r, \ldots, n+1, j = 1, \ldots, r+1$. The dimension of P is equal to $n^2 - 1 - (n-r)(r+1)$.

Next we define a surjective regular map of algebraic k-sets $f : GL(n+1, K) \to G(r+1, n+1)$ by the formula $M \to M(L_0)$, where $L_0 = Ke_1 + \ldots + Ke_{r+1}$. If $M = (a_{ij})$, then

$$M(L_0) = \text{span}\{a_{11}e_1 + \ldots + a_{n+11}e_{n+1}, \ldots, a_{1r+1}e_1 + \ldots + a_{n+1r+1}\}$$

so that the Plücker coordinates $p[i_1, \ldots, i_{r+1}]$ of $M(L_0)$ are equal to the minor of the matrix (a_{ij}) formed by the first $r+1$ columns and the rows indexed by the set $\{i_1, \ldots, i_{r+1}\}$. This shows that f is a regular map from the affine k-set $GL(n+1, K)$ to the projective algebraic k-set $G(r+1, n+1)$. Its fibres are isomorphic to P. By the theorem on the dimension of fibres from Lecture 11, we obtain that

$$\dim G(r+1, n+1) = \dim GL(n+1, K) - \dim P = n^2 - 1 - (n^2 - 1 - (n-r)(r+1)) = (n-r)(r+1).$$

Since $GL(n+1, K)$ is irreducible, $G(r+1, n+1)$ is irreducible.

Now let us give another proof of this result. Choose the Plücker coordinates $p[j_1, \ldots, j_{r+1}]$ and consider the open subsets $D(p[j_1, \ldots, j_{r+1}]) \subset \mathbf{P}_k^{\binom{n+1}{r+1}-1}(K)$. The intersection $D(p[j_1, \ldots, j_{r+1}]) \cap G(r+1, n+1)$ is equal to the set of linear subspaces L which admit a basis

$$f_1 = a_{11}e_1 + \ldots + a_{1n+1}e_n, \ldots, \; f_{r+1} = a_{r+11}e_1 + \ldots + a_{r+1n+1}e_n,$$

such that $p[j_1, \ldots, j_{r+1}] = \det(A_{j_1 j_2 \ldots j_{r+1}}) \neq 0$, where

$$A_{i_1 i_2 \ldots i_{r+1}} = \begin{pmatrix} a_{1j_1} & a_{1j_2} & \ldots & a_{1j_{r+1}} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{r+1j_1} & a_{1j_2} & \ldots & a_{r+1j_{r+1}} \end{pmatrix}$$

After we replace $f_1, \ldots, f_{r+1}$ with $f'_1, \ldots, f'_{r+1}$ such that

$$f'_1 = b_{11}f_1 + \ldots + b_{1r+1}f_{r+1}, \ldots, f'_{r+1} = b_{r+11}f_1 + \ldots + b_{r+1r+1}f_{r+1},$$

where (b_{ij}) is the inverse of the matrix $A_{j_1 j_2 \ldots j_{r+1}}$, we may assume that $A_{i_1 i_2 \ldots i_{r+1}}$ is the identity matrix I_{r+1}. Then we may take all $(n-r)(r+1)$ other entries $a_{ij}, j \neq j_k$ arbitrary, and obtain that $D(p[i_1, \ldots, i_{r+1}]) \cap G(r+1, n+1)$ is isomorphic to the affine space $\mathbf{A}_k^{(n-r)(r+1)}(K)$. Thus $G(r+1, n+1)$ is covered by $\binom{n+1}{r+1}$ open subsets isomorphic to the affine space of dimension $(n-r)(r+1)$. This obviously proves the assertion.

Example. Let us consider the case $r = 1, n = 3$. Then $G(2,4) \subset \mathbf{P}^5$ parametrizes lines in $\mathbf{P}^3(K)$. We have six Plücker coordinates $p[ij], i, j = 1, 2, 3, 4$. The condition that $\omega = \sum p[ij]e_i \wedge e_j$ is a decomposable 2-vector $\omega \wedge \omega = 0$. In fact, obviously all decomposable 2-vectors satisfy this condition. On the other hand, this condition represents an irreducible quadric hypersurface with the equation:

$$(\sum p[ij]e_i \wedge e_j) \wedge (\sum p[ij]e_i \wedge e_j) = p[12]p[34] + p[13]p[24] + p[14]p[23] = 0.$$

Since we know already that $G(2,4)$ is an irreducible projective set of dimension 4, we obtain that it coincides with the quadric $V(Q)$ where $Q = p[12]p[34] + p[13]p[24] + p[14]p[23]$. Evidently Q is a non-degenerate quadratic form.

Remark 1. The equation for $G(2,4)$ given in the proof of Theorem 1 differs from the equation $Q=0$ by a factor. Any Grassmannian $G(r+1,n+1)$ can be given by a system of equations of degree 2, so called the *Plücker equations.* They look as follows:

$$\sum_{s=1}^{r+2}(-1)^s p[i_1,\ldots,i_r,j_s]p[j_1\ldots,j_{s-1},j_{s+1},\ldots,j_{r+2}]=0,$$

where $\{i_1,\ldots,i_r\}$ and $\{j_1,\ldots,j_{r+2}\}$ are any two strictly increasing sequences of the set $\{1,\ldots,n+1\}$.

We denote by $\mathrm{Hyp}(d,n)$ the projective space $\mathbf{P}^{\binom{n+d}{d}-1}$. If we use $\Upsilon_{i_0,\ldots,i_n}, 0\le i_j, i_0+\ldots+i_n=d$ to denote projective coordinates in this space then each K-point $(\ldots,a_{i_0,\ldots,i_n},\ldots)$ of this space defines the projective K-subvariety $F=0$ of $\mathbf{P}^n_K$ where

$$F=\sum_{i_0,\ldots,i_n} a_{i_0,\ldots,i_n}T_0^{i_0}\cdots T_n^{i_n}=0.$$

Thus we can view K-points of the projective space $\mathbf{P}^{\binom{n+d}{d}-1}$ as projective hypersurfaces of degree d. This explains the notation. In the special case when $d=1$, the space $\mathrm{Hyp}(1,n)$ is called the *dual space* of $\mathbf{P}^n_k$ and is denoted by $\check{\mathbf{P}}^n_k$. Its K-points are in a bijective correspondence with linear subspaces of $\mathbf{P}^n_k(K)$ of dimension $n-1$ (*hyperplanes*) .

Now, everything is ready to solve our problem. Fix any algebraically closed field K. Let $H=\mathrm{Hyp}(d,n)(K)$ and $G=G(r+1,n+1)(K)$. Define

$$I(r,d,n)(K)=\{(X,E)\in H(K)\times G(K): E\subset X\}.$$

Lemma 1. *$I(r,d,n)$ is a closed irreducible subset of $H\times G$ of dimension equal to $(r+1)(n-r)+\binom{n+d}{d}-\binom{r+d}{d}-1$.*

Proof. Let E' denote the linear subspace of K^{n+1} corresponding to E. Let $f_1,\ldots,f_{r+1}$ be a basis of E', extended to a basis $(f_1,\ldots,f_{n+1})$ of K^{n+1}. Any $x\in E'$ defines a linear form ϕ_x on $\bigwedge^r(K^{n+1})$ given by the formula

$$x\wedge\omega\wedge f_{r+2}\wedge\ldots\wedge f_{n+1}=\phi_x(\omega)f_1\wedge\ldots\wedge f_{n+1}.$$

In particular, if $x=\lambda_1 f_1+\ldots+\lambda_{r+1}f_{r+1}$, then taking the wedge product of both sides with each $f_1\wedge\ldots\wedge f_{i-1}\wedge f_{i+1}\wedge\ldots f_{n+1}$, we obtain

$$x=\phi_x(f_2\wedge\ldots\wedge f_{r+1})f_1-\phi_x(f_1\wedge f_3\wedge\ldots\wedge f_{r+1})f_2+\ldots+(-1)^r\phi_x(f_1\wedge\ldots\wedge f_r)f_{r+1}.$$

Let $(e_1^*,\ldots,e_{n+1}^*)$ be the dual basis of the canonical basis of K^{n+1}. Writing $\phi_x=\sum\alpha_{i_1\ldots i_r}e_{i_1}^*\wedge\ldots\wedge e_{i_r}^*$, we get that λ_i are equal to some linear combinations of the Plücker coordinates of E'. Now plugging in $(\lambda_0,\ldots,\lambda_n)$ into the equation of X , we see that $I(r,d,n)$ is given by bi-homogeneous polynomials in the coefficients of F and in the Plücker coordinates of E. This proves that $I(r,d,n)$ is a closed subset of the product $H\times G$. Now consider the projection $p: I(r,d,n)\to G$. For each $E\in G$, the fibre $p^{-1}(E)$ consists of all hypersurfaces $V(F)$ containing E. Choose a coordinate system such that E is given by the equations $T_{r+1}=\ldots=T_n=0$. Then $E\subset V(F)$ if and only if each monomial entering into F with non-zero coefficient contains some positive power of T_i with

$i \geq r+1$. In other words F is defined by vanishing of all coefficients at the monomials of degree d in the variables $T_0, \ldots, T_r$. This gives $\binom{r+d}{r}$ linear conditions on the coefficients of F, hence $\dim p^{-1}(E) = \binom{n+d}{d} - 1 - \binom{r+d}{r}$. Let us assume that $I(r,d,n)$ is irreducible. By the Theorem on dimension of fibres,

$$\dim I(r,d,n) = \dim p^{-1}(E) + \dim G = (n-r)(r+1) + \binom{n+d}{d} - 1 - \binom{r+d}{r}.$$

It remains to prove the irreducibility of $I(r,d,n)$. Considering the projection $p : I(r,d,n) \to G$, the assertion follows from the following:

Lemma 2. *Let $f : X \to Y$ be a surjective regular map of projective algebraic sets. Assume that Y is irreducible and all fibres of f are irreducible and of the same dimension n. Then X is irreducible.*

Proof. Let $X = X_1 \cup \ldots \cup X_n$ be the union of irreducible closed sets. Since f is a map of projective sets, the images $f(X_i)$ are closed and irreducible. By assumption, Y is irreducible, hence the set $I = \{i : f(X_i) = Y\}$ is not empty. For every $y \in Y \setminus (\cup_{i \notin I} f(X_i))$, we have $f^{-1}(y) = \cup_{i \in I}(X_i \cap f^{-1}(y))$. Since $f^{-1}(y)$ is irreducible, there exists $X_i, i \in I$, such that $f^{-1}(y) \subset X_i$. Since the set I is finite, we can find an open subset $U \subset Y$ such that $f^{-1}(y) \subset X_i$ for all $y \in U$. Let $f_i : X_i \to Y$ be the restriction of f to X_i. By theorem on dimension of fibres, any fibre of f_i is of dimension $\geq n$. By assumption, $\dim f_i^{-1}(y) \geq n = \dim f^{-1}(y)$. This implies that $f_i^{-1}(y) = f^{-1}(y)$ for any $y \in Y$. This certainly implies that $X_i = X$ proving the assertion.

Theorem 2. *Assume that*

$$(n-r)(r+1) < \binom{r+d}{r}.$$

Then the subset of $\mathrm{Hyp}(d,n)(K)$ which consists of hypersurfaces containing a linear subspace of dimension r is a proper closed subset.

Proof. Consider the other projection $q : I(r,d,n) \to H = \mathrm{Hyp}(d,n)(K)$. Since $I(r,d,n)$ is a projective set, its image is a closed subset of H. Suppose q is surjective. Then

$$(n-r)(r+1) + \binom{n+d}{d} - 1 - \binom{r+d}{r} = \dim I(r,d,n) \geq \dim H = \binom{n+d}{d} - 1.$$

This is impossible in view of the assumption of the theorem.

Remark 2. One expects that each $V(F) \in \mathrm{Hyp}(d,n)$ contains a linear subspace of dimension r when $(n-r)(r+1) \geq \binom{r+d}{r}$. This is true if $d > 2$ but false if $d = 2$. For example let $d = 2, n = 4$. A nonsingular quadratic form in 5 variables does not contain isotropic subspaces of dimension 3. Hence the corresponding quadric does not contain planes. However, $(n-r)(r+1) = 6 \geq \binom{r+d}{r} = 6$.

From now on we restrict ourselves with the case $n = 3$ and $d = 3$, i.e. cubic surfaces in $\mathbf{P}^3(K)$. We shall be looking for lines on cubic surfaces. In this case $(n-r)(r+1) = 6 > \binom{r+d}{d} = 4$, so we expect that every cubic surface has a line. As we saw in the previous remark it needs to be proven.

Theorem 3. *(i) Every cubic surface X contains a line.*

(ii) There exists an open subset $U \subset \mathrm{Hyp}(3,3)(K)$ such that any $X \in U$ contains exactly 27 lines.

Proof. (i) In the notation of the proof of Theorem 2, it suffices to show that the projection map $q : I(1,3,3) \to \mathrm{Hyp}(3,3)(K)$ is surjective. Suppose the image of q is a proper closed subset Y of $\mathrm{Hyp}(3,3)(K)$. Then $\dim Y < \dim \mathrm{Hyp}(3,3)(K) = 19$ and $\dim I(1,3,3) = 19$. By the theorem on dimension of fibres, we obtain that all fibres of q are of dimension at least one. In particular, every cubic surface containing a line contains infinitely many of them. But let us consider the surface X given by the equation

$$T_1T_2T_3 - T_0^3 = 0.$$

Suppose a line ℓ lies on X. Let $(a_0, a_1, a_2, a_3) \in \ell$. If $a_0 \neq 0$, then $a_i \neq 0, i \neq 0$. On the other hand, every line hits the planes $T_i = 0$. This shows that ℓ is contained in the plane $T_0 = 0$. But there are only three lines on X contained in this plane: $T_i = T_0 = 0, i = 1, 2$ and 3. Therefore X contains only 3 lines. This proves the first assertion.

(ii) We already know that every cubic surface $X = V(F)$ has at least one line. Pick up such a line ℓ. Without loss of generality we may assume that it is given by the equation:

$$T_2 = T_3 = 0.$$

As we saw in the proof of Lemma 1:

$$F = T_2Q_0(T_0, T_1, T_2, T_3) + T_3Q_1(T_0, T_1, T_2, T_3) = 0,$$

where Q_0 and Q_1 are quadratic homogeneous polynomials. Each plane π containg the line ℓ is given by the equation

$$\lambda T_2 - \mu T_3 = 0$$

for some scalars $\lambda, \mu \in K$. The intersection $\pi \cap V(F)$ contains the line λ and a curve of degree 2 in π. More explicitly, choose coordinates t_0, t_1, t_2 in the plane, related to our coordinates T_0, T_1, T_2, T_3 by the formulas:

$$T_0 = t_0, T_1 = t_1, T_2 = \mu t_2, T_3 = \lambda t_2.$$

Plugging these expression into F, we obtain:

$$\mu t_2 Q_0(t_0, t_1, \mu t_2, \lambda t_2) + \lambda t_2 Q_1(t_0, t_1, \mu t_2, \lambda t_3) = 0.$$

This shows that $\pi \cap X \subset \pi$ consists of the line ℓ with the equation $t_2 = 0$ and the conic $C(\lambda, \mu)$ with the equation:

$$\mu Q_0(t_0, t_1, \mu t_2, \lambda t_2) + \lambda Q_1(t_0, t_1, \mu t_2, \lambda t_2) = 0.$$

We may also assume that the line enters with multiplicity one (since we take "general " coeficients of F). Let

$$Q_0 = \sum_{0 \leq i < j \leq 3} a_{ij}T_iT_j, \quad Q_1 = \sum_{0 \leq i < j \leq 3} b_{ij}T_iT_j.$$

Then $C(\lambda, \mu)$ is given by the equation:

$$(\mu a_{00} + \lambda b_{00})t_0^2 + (\mu a_{11} + \lambda b_{11})t_1^2 + (\mu^2(\mu a_{22} + \lambda b_{22}) + \lambda^2(\mu a_{33} + \lambda b_{33}))t_2^2 + (\mu a_{01} + \lambda b_{01})t_0t_1$$

$$+(\mu(\mu a_{02} + \lambda b_{02}) + \lambda(\mu a_{03} + \lambda b_{03}))t_0t_2 + (\mu^2 a_{12} + \lambda\mu b_{12}) + (\mu\lambda a_{13} + \lambda^2 b_{13}))t_1t_2 = 0.$$

Now, let us start vary the parameters λ and μ and see how many reducible conics $C(\lambda,\mu)$ we obtain. The conic $C(\lambda,\mu)$ is reducible if and only if the quadratic form defining it is degenerate. The condition for the latter is the vanishing of the discriminant D of the quadratic form $C(\lambda,\mu)$. Observe that D is a homogeneous polynomial of degree 5 in λ,μ. Thus there exists a Zariski open subset of $\mathrm{Hyp}(3,3)$ for which this determinant has 5 distinct roots (λ_i,μ_i). Each such solution defines a plane π_i which cut out on X the line ℓ and a reducible conic. The latter is the union of two lines or a double line. Again, for some open subset of $\mathrm{Hyp}(3,3)$ we expect that the double line case does not occur. Thus we found 11 lines on X: the line ℓ and 5 pairs of lines ℓ_i,ℓ_i' lying each lying in the plane π_i. Pick up some plane, say π_1. We have 3 lines ℓ,ℓ', and ℓ'' in π_1. Replacing ℓ by ℓ', and then by ℓ'', and repeating the construction, we obtain 4 planes through ℓ,ℓ' and 4 planes through ℓ'', each containing a pair of lines. Altogether we found $3+8+8+8=27$ lines on X. To see that all lines are accounted for, we observe that any line intersecting either ℓ, or ℓ', or ℓ'' lies in one of the planes we have considered before. So it has been accounted for. Now let L be any line. We find a plane π through L that contains three lines L,L' and L'' on X. This plane intersects the lines ℓ,ℓ', and ℓ'' at some points p,p' and p'' respectively. We may assume that these points are distinct. Otherwise we find three nocoplanar lines in X passing through one point. As we shall see later this implies that X is singular at this point. Since neither L' nor L'' can pass through two of these points, one of these points lie on L. Hence L is coplanar with one of the lines ℓ,ℓ',ℓ''. Therefore L has been accounted for.

Remark 3. Using more techniques one can show that every "nonsingular" (in the sense of the next lectures) cubic surface contains exactly 27 lines. Let us define the graph whose vertices are the lines and two vertices are joined by an edge if the lines intersect. This graph is independent on the choice of a nonsingular cubic surface and its group of symmetries is isomorphic to the group $W(E_6)$ of order 51840 (the Weyl group of the root system of a simple Lie algebra of type E_6).

Exercises.

1. Show that the set π_x of lines in $\mathbf{P}^3(K)$ passing through a point $x\in\mathbf{P}^3(K)$ is a closed subset of $G(2,4)$ isomorphic to $\mathbf{P}^2(K)$. Also show that the set π_P of lines in $\mathbf{P}^3(K)$ contained in a plane $P\subset\mathbf{P}^3(K)$ is a closed subset of $G(2,4)$ isomorphic to $\mathbf{P}^2(K)$.
2. Prove that the subset of quartic surfaces in $\mathrm{Hyp}(4,3)$ which contain a line is an irreducible closed subset of $\mathrm{Hyp}(4,3)$ of codimension 1.
3. Prove that every hypersurface of degree $d\le 5$ in $\mathbf{P}^4(K)$ contains a line, and, if $d\le 4$, then it contains infinitely many lines.
4. Let X be a general cubic hypersurface in $\mathbf{P}^4(K)$ (general means that X belongs to an open subset U of $\mathrm{Hyp}(3,4)$). Show that there exists an open subset $V\subset X$ such that any $x\in V$ lies on exactly six lines contained in X.

5*. Let $1\le n_1<n_2<\ldots<n_r\le n+1$, a flag in K^{n+1} of type $(n_1,\ldots,n_r)$ is a chain of linear subspaces $L_1\subset\ldots\subset L_r$ with $\dim L_i=n_i$.

(a) Show that there exists a projective algebraic variety over a field k such that its restriction to the category of field extensions of k is equal to the functor $K\to\{$flags in K^{n+1} of type $(n_1,\ldots,n_r)\}$. This variety is called the *flag variety* of type $(n_1,\ldots,n_r)$ and is denoted by $F_k(n_1,\ldots,n_r;n+1)$.

(b) Find the dimension of $F_k(n_1,\ldots,n_r;n)$.

(c) Show that the functor $\{$Field-extensions of $k\}\to(Sets)$, $K\to F(1,2;3)(K)$ is isomorphic to the functor $K\to\{$rank 1 matrices of size 3×3 with zero trace and coefficients in $K\}/K^*$.

6. By analyzing the proof of Theorem 3 show the following:

(a) The set of 27 lines on a cubic surface X contains 45 triples of lines which lie in a plane (called a *tritangent plane*).
(b) There exist 12 lines $l_1, \ldots, l_6, l'_1, \ldots, l'_6$ such that $l'_i \cap l_i = \emptyset, i = 1, \ldots, 6, l_i \cap l'_j \neq \emptyset$ if $i \neq j$. Such a set is called a *double-sixer.*
(c)* Show that there are 36 different double sixers.
(d) Check all the previous assertions for the Fermat cubic.

7*. Prove that
(a) Each nonsingular cubic surface F contains 9 lines $\ell_{ij}, i, j = 1, 2, 3$ such that $\ell_{ij} \cap \ell_{km} \neq \emptyset$ if and only if $i = k$ or $j = m$.
(b) Using (a) show that the cubic surface can be given by the equation

$$\det \begin{pmatrix} L_1 & 0 & L'_1 \\ L'_2 & L_2 & 0 \\ 0 & L'_3 & L_3 \end{pmatrix} = 0,$$

where L_i, L'_i are linear forms.
(c) Show that the map $T : F \to \mathbf{P}^2$ which assigns to a point $x \in F$ the set of solutions of the equation $(t_0, t_1, t_2) \cdot A = 0$ is a birational map. Here A is the matrix of linear from from (b).
(d) Find an explicit formulas for the inverse birational map T^{-1}.

8. Using Problem 5 (b),(c) show that the group W of symmetries of 27 lines consists of 51840 elements.

9*. Let $k = k_0(t)$, where k_0 is an algebraically closed field and $F(T_0, \ldots, T_n) \in k[T_1, \ldots, T_n]$ be a homogeneous polynomial of degree $d < n$. Show that $V(F)(k) \neq \emptyset$ (Tsen's Theorem).

Lecture 13. TANGENT SPACE

The notion of the tangent space is familiar from analytic geometry. For example, let $F(x, y) = 0$ be a curve in $\mathbf{R}^2$ and let $a = (x_0, y_0)$ be a point lying on this surface. The tangent line of X at the point a is defined by the equation:

$$\frac{\partial F}{\partial x}(a)(x - x_0) + \frac{\partial F}{\partial y}(a)(y - y_0) = 0.$$

It is defined only if at least one partial derivative of F at a is not equal to zero. In this case the point is called nonsingular. Otherwise it is said to be singular.

Another notion of the tangent space is familiar from the theory of differentiable manifolds. Let X be a differentiable manifold and a be its point. By definition, a tangent vector t_a of X at a is a derivation (or differentiation) of the ring $\mathcal{O}(X)$ of differentiable functions on X, that is, a $\mathbf{R}$-linear map $\delta : \mathcal{O}(X) \to \mathbf{R}$ such that

$$\delta(fg) = f(a)\delta(g) + g(a)\delta(f) \quad \text{for any } f, g \in \mathcal{O}(X).$$

It is defined by derivation of a function f along t_a given by the formula

$$< f, t_a >= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) t_i$$

where $(t_1, \ldots, t_n)$ are the coordinates of t_a and $(x_1, \ldots, x_n)$ are the local coordinates of X at the point a.

In this lecture we introduce and study the notion of a tangent space and a nonsingular point for arbitrary algebraic sets or varieties.

For every k-algebra K let $K[\varepsilon] = K[t]/(t^2)$ be the K-algebra of dual numbers. If ε is taken to be $t\text{mod}(t^2)$, then $K[\varepsilon]$ consists of linear combinations $a + b\varepsilon, a, b \in K$, which are added coordinate wise and multiplied by the rule

$$(a + b\varepsilon)(a' + b'\varepsilon) = aa' + (ab' + a'b)\varepsilon.$$

We denote by

$$\alpha_1 : K[\varepsilon] \to K$$

the natural homomorphism $a + b\varepsilon \to a$. Its kernel is the ideal $(\varepsilon) = \{b\varepsilon, b \in K\}$.

Definition. Let F be any functor from the category of k-algebras to the category of sets. Let $x \in F(K)$ be a "K-point" of F. A *tangent vector* t_x of F at x is a $K[\varepsilon]$-point $t_x \in F(K[\varepsilon])$ such that $F(\alpha_1)(t_x) = x$. The set of tangent vectors of F at x is denoted by $T(F)_x$ and is called the *tangent space* of F at x.

Example 1. Assume X is an affine algebraic variety given by a system of equations

$$F_1(Z_1,\ldots,Z_n) = 0,\ldots,F_r(Z_1,\ldots,Z_n) = 0.$$

A point $x \in X(K)$ is a solution $(a_1,\ldots,a_n) \in K^n$ of this system. A tangent vector t_x is a solution $(a_1+b_1\varepsilon,\ldots,a_n+b_n\varepsilon) \in X(K[\varepsilon])$ of the same system. Write down the polynomials $F_i(Z)$ in the form :

$$F_i(Z_1,\ldots,Z_n) = G_i(Z_1-a_1,\ldots,Z_n-a_n) = \sum_{j=1}^{n}\alpha_j^{(i)}(Z_i-a_i) + \sum_{j,k=1}^{n}\alpha_{jk}^{(i)}(Z_j-a_i)(Z_k-a_j)+\ldots.$$

(Taylor's expansion). Note that G_i do not contain the constant term because $(a_1,\ldots,a_n) \in X(K)$. By definition, the coefficient $\alpha_j^{(i)}$ is the partial derivative of F_i with respect to Z_j at the point $x = (a_1,\ldots,a_n)$. It is denoted by $\frac{\partial F_i}{\partial Z_j}(x)$. Obviously it is an element of K. Now we plug the point $(a_1+b_1\varepsilon,\ldots,a_n+b_n\varepsilon)$ into the previous equations to obtain

$$F_i(a_1+b_1\varepsilon,\ldots,a_n+b_n\varepsilon) = \sum_{j=1}^{n}\alpha_j^{(i)}b_i\varepsilon + \sum_{j,k=1}^{n}\alpha_{jk}^{(i)}b_ib_j\varepsilon^2 + (\ldots)\varepsilon^3+\ldots = \sum_{j=1}^{n}\alpha_j^{(i)}b_i\varepsilon = 0.$$

From this we deduce that $(b_1,\ldots,b_n)$ satisfies the system of linear homogeneous equations:

$$(*) \qquad \sum_{j=1}^{n}\frac{\partial F_i}{\partial Z_j}(x)b_j = 0, \quad i = 1,\ldots,r.$$

Thus the set of tangent vectors $T(X)_x$ is bijective to the linear space of solutions of $(*)$. In particular, we have introduced the structure of a K-module on $T(X)_x$.

Example 2. Assume $F = \mathbf{P}_k^n$ is projective space over k. Let $x \in \mathbf{P}_k^n(K)$ be given by a global line $M_0 = K(a_0,\ldots,a_n) \subset K^{n+1}$. A tangent vector at x is a local line M over $K[\varepsilon]$ such that $M/\varepsilon M \cong M_0$. It is easy to see that any lift $(a_0+\varepsilon t_0,\ldots,a_n+\varepsilon t_n)$ of the generator of M_0 generates M. This will follow from the Nakayama Lemma which we shall prove in the next Lecture. Note that since $1 = \sum_i b_ia_i$ for some $b_i \in K$ we obtain $\sum_i b_i(a_i+\varepsilon t_i) = 1+\varepsilon(\sum_i b_it_i) \in K[\varepsilon]^*$, hence $K[\varepsilon](a_0+\varepsilon t_0,\ldots,a_n+\varepsilon t_n)$ is a global line for any $(t_0,\ldots,t_n)$. This shows that $M \in T(\mathbf{P}_k^n)_x$ is determined by $(t_0,\ldots,t_n)$ up to the equivalence relation defined by

$$(t_0,\ldots,t_n) \sim (t'_0,\ldots,t'_n) \quad \text{if } (a_0+\varepsilon t_0,\ldots,a_n+\varepsilon t_n) = (a_0+\varepsilon t'_0,\ldots,a_n+\varepsilon t'_n) \text{ in } \mathbf{P}_k^n(K[\varepsilon]).$$

The latter means that

$$(a_0+\varepsilon t'_0,\ldots,a_n+\varepsilon t'_n) = (a+\varepsilon b)(a_0+\varepsilon t_0,\ldots,a_n+\varepsilon t_n)$$

for some $a+b\varepsilon \in K[\varepsilon]^*$. This immediately implies that $a = 1, b \in K$ is arbitrary and $(t'_0,\ldots,t'_n) = (t_0,\ldots,t_n)+b(a_0,\ldots,a_n)$. Thus $T(\mathbf{P}_k^n)_x$ can be identified with K^{n+1}/M_0, or better, since we were fixing a representative $(a_0,\ldots,a_n)$ of M_0 with $\mathrm{Hom}(M_0,K^{n+1}/M_0)$. This has also a natural structure of a K-module.

In both of these examples we see that $T(X)_x$ has a structure of a K-module. However, in the first example it is not clear that this structure is independent of a choice of the system of equations defining X. To overcome this difficulty, we shall give another, more invariant, definition of $T(X)_x$.

Let A be a commutative k-algebra and let M be A-module. A M-derivation of A is a linear map of the corresponding k-linear spaces $\delta : A \to M$ such that for all $a, b \in A$

$$\delta(ab) = a\delta(b) + b\delta(a).$$

The set of M-derivations is denoted by $Der_k(A, M)$. It has a natural structure of a A-module via

$$(a\delta)(b) = a\delta(b) \quad \text{for all } a, b \in A.$$

We will be interested in a special case of this definition.

Lemma 1. *If $f : A \to B$ is a homomorphism of k-algebras, and $\delta : B \to M$ is a M-derivation of B, then the composition $\delta \circ f : A \to B \to M$ is a $M_{[f]}$-derivation of A, where $M_{[f]}$ is the A-module obtained from M by the operation of restriction of scalars (i.e., $a \cdot m = f(a)m$ for any $a \in A, m \in M$).*

Proof. Trivial verification of the definition.

Let us apply this to our situation. Note that the k-linear map:

$$\alpha_2 : K[\varepsilon] \to K, a + b\varepsilon \to b$$

is a K-derivation of $K[\varepsilon]$ considered as a K-algebra. Here K is considered as a $K[\varepsilon]$-module by means of the homomorphism $\phi_1 : K[\varepsilon] \to K, a + b\varepsilon \mapsto a$. We identify a K-point $x \in X(K)$ with a homomorphism of k-algebras $x : \mathcal{O}(X) \to K$, $\phi \mapsto \phi(x)$. A tangent vector $t_x \in T(X)_x$ is identified with a homomorphism $t_x : \mathcal{O}(X) \to K[\varepsilon]$. Its composition with the derivation $\alpha_2 : K[\varepsilon] \to K, a + b\varepsilon \mapsto b$, is a K-derivation of $k[X]$. Here K is considered as a $\mathcal{O}(X)$-module via the homomorphism x. This defines a map:

$$T(X)_x \to Der_k(\mathcal{O}(X), K)_x$$

where the subscript x stands to remind us about the structure of a $\mathcal{O}(X)$-module on K. By definition,

$$Der_k(\mathcal{O}(X), K)_x = \{\delta \in Hom_k(\mathcal{O}(X), K) : \delta(pq) = p(x)\delta(q) + q(x)\delta(p) \quad \text{for any } p, q \in \mathcal{O}(X)\}.$$

Lemma 2. *The map*

$$T(X)_x \to Der_k(\mathcal{O}(X), K)_x$$

is a bijection.

Proof. Let $\delta \in Der_k(k[X], K)_x$. We define a map $f_\delta : \mathcal{O}(X) \to K[\varepsilon]$ by the formula:

$$f_\delta(p) = p(x) + \varepsilon\delta(p).$$

It is easy to verify that f_δ is a homomorphism, and its composition with $\alpha_1 : K[\varepsilon] \to K$ is equal to x. Thus f_δ defines a tangent vector at x and the formula $\delta \mapsto f_\delta$ makes the inverse of our map.

Now $Der_k(\mathcal{O}(X), K)_x$ has a structure of a K-module, defined by the formula $(a\delta)(p) = a\delta(p)$ for any $a \in K, p \in \mathcal{O}(X)$. We transfer this structure to $T(X)_x$ by means of the bijection from Lemma 2. This structure of a K-module on $T(X)_x$ is obviously independent of the choice of equations defining X. We leave to the reader to verify that this structure agrees with the one defined in the beginning of the lecture.

Let us specialize our definition to the case when $K = k$. For any $x \in X(k)$ (*rational point* of X) the kernel of the homomorphism $x : \mathcal{O}(X) \to k$ is a maximal ideal m_x of $\mathcal{O}(X)$ and $\mathcal{O}(X)/m_x \cong k$. Let $\delta \in Der_k(\mathcal{O}(X), k)$ be a k-derivation of $\mathcal{O}(X)$. For any $p, q \in m_x$, we have

$$\delta(p \cdot q) = p(x)\delta(q) + q(x)\delta(p) = 0.$$

Thus the restriction of δ to m_x^2 is identical zero.

Lemma 3. *The restriction map $\delta \to \delta|m_x$, defines an isomorphism of k-linear spaces*

$$T(X)_x \to Hom_k(m_x/m_x^2, k).$$

Proof. Since $\mathcal{O}(X)/m_x^2$ has a natural structure of a k-algebra there is a canonical homomorphism $k \to \mathcal{O}(X)/m_x^2$ such that its composition with the factor map $\mathcal{O}(X)/m_x^2 \to \mathcal{O}(X)/m_x = k$ is the identity. We shall identify k with the subring of $\mathcal{O}(X)/m_x^2$ by means of this map so that the restriction of the factor map $\mathcal{O}(X)/m_x^2 \to \mathcal{O}(X)/m_x$ to k is the identity. For any $p \in \mathcal{O}(X)$ we denote by p_x the residue of p mod m_x^2. Obviously, $p_x - p(x) \in m_x/m_x^2$, so that for every linear function $f \in Hom_k(m_x/m_x^2, k)$ by setting for any $p \in \mathcal{O}(X)$

$$\delta(p) = f(p_x - p(x)).$$

Since for any $p, q \in \mathcal{O}(X), (p_x - p(x))(q_x - q(x)) \in m_x^2$, we have

$$\delta(pq) = f(p_xq_x - p(x)q(x)) = f((p_x - p(x))(q_x - q(x)) + p(x)(q_x - q(x)) + q(x)(p_x - p(x))) =$$

$$= f(p(x)(q_x - q(x)) + q(x)(p_x - p(x))) = p(x)f(q_x - q(x)) + q(x)f(p_x - p(x))) = p(x)\delta(q) + q(x)\delta(p).$$

We leave to the reader to verify that the constructed map $f \mapsto \delta$ is the needed inverse.

Let $f : F \to G$ be a morphism of two functors from $k - Alg$ to $Sets$, let $x \in F(K)$, and $y = f(K)(x) \in G(K)$. By functoriality, the map $f(K[\varepsilon]) : F(K[\varepsilon]) \to G(K[\varepsilon])$ induces a natural map

$$(df)_x : T(F)_x \to T(G)_y.$$

It is called the *differential* of f at the point x. If $f : X \to Y$ is a morphism of affine k-varieties corresponding to a homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ of k-algebras, $x \in X(k), y = f(k)(x) \in Y(k)$, then it is immediately verified that the differential $(df)_x$ coincides with the transpose of the linear map $m_y/m_y^2 \to m_x/m_x^2$ induced by the homomorphism $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$.

Proposition 1 (Chain Rule). *Let $f : F \to G, g : G \to H$ be morphisms of functors from k-Alg to Sets, $x \in F(K), y = f(K)(x) \in G(K)$. Then*

$$d(g \circ f)_x = (dg)_y \circ (df)_x.$$

Proof. Immediately follows from the definition of functors.

Now we can define the tangent space for any algebraic set $V \subset \mathbf{P}^n(K)$. Here K is a fixed algebraically closed field. First, we assume that V is affine. Choose the unique affine algebraic K-variety X such that $I(X)$ is radical and $X(K) = V$. Then we define the Zariski tangent space $T(V)_x$ of V at x by setting

$$T(V)_x = T(X)_x.$$

By Lemma 3, for every $x \in V$ we have an isomorphism of K-linear spaces.

$$T(X)_x \cong Hom_K(m_x/m_x^2, K).$$

Lemma 4. *Let X be an affine irreducible algebraic set, $x \in X$, and let U be an open affine subset of X which contains x. The differential of the inclusion map $j : U \hookrightarrow X$ at the point x induces an isomorphism of the tangent spaces $T(U)_x \to T(X)_x$.*

Proof. Let $V = D(\phi)$ be a principal open affine neighborhood of x contained in U. Applying Proposition 1 to the composition of the inclusions $V \hookrightarrow U \hookrightarrow X$, we see that we may assume that $U = V$. If X is given by equations $F_j(Z) = 0, j = 1, \ldots, r$, in $\mathbf{A}(K)^n$, then U is isomorphic to a closed subset of $\mathbf{A}^{n+1}(K)$ given by the equations:

$$F_j(Z_1, \ldots, Z_n) = 0, j = 1, \ldots, r, \quad G(Z_1, \ldots, Z_n)Z_{n+1} - 1 = 0$$

where $\phi = G(Z_1, \ldots, Z_n) \bmod I(X)$. Let $x = (a_1, \ldots, a_n) \in X$ and $(a_1, \ldots, a_n, G(x)^{-1})$ be the corresponding point of U. Computing the tangent spaces, we obtain that $T(X)_x$ is isomorphic to the space of solutions of the system

$$\sum_{i=1}^{n} \frac{\partial F_j}{\partial Z_i}(x) b_i = 0, j = 1, \ldots, r$$

and $T(U)_x$ is isomorphic to the space of solutions of the system

$$\sum_{i=1}^{n} \frac{\partial F_j}{\partial Z_i}(x) b_i = 0, j = 1, \ldots, r, \sum_{i=1}^{n} \frac{\partial G}{\partial Z_i}(x) G(x)^{-1} b_i + G(x) b_{n+1} = 0.$$

Under the projection $(b_1, \ldots, b_{n+1}) \mapsto (b_1, \ldots, b_n)$ the spaces of solutions are isomorphic. It is easy to see that this projection is equal to the differential $(dj)_x$ (check it !).

It follows from the previous lemma that the notion of the tangent space $T(X)_x$ is local, i.e. it can be defined by replacing the whole set X by any affine open neighborhood of x. Using this we can define the Zariski tangent space $T(X)_x$ of any algebraic set X at its point x by setting

$$T(X)_x = T(U)_x$$

where U is any affine open neighborhood of the point x. It follows from the above that this definition is independent (up to natural isomorphism) of the choice of U.

One can give even more "local" definition of $T(X)_x$ by introducing the notion of the local ring of X at a point $x \in X$.

Definition. The *local ring* of X at $x \in X$ is the factor set

$$\mathcal{O}_{X,x} = \bigcup_{x \in U} \mathcal{O}(U)/R$$

where U runs through the set of all open affine neighbourhoods of x and the equivalence relation R is defined as follows:

Let $f \in \mathcal{O}(U), g \in \mathcal{O}(V)$, then

$$f \equiv g \Longleftrightarrow f|W = g|W \quad \text{for some open affine neighborhood of } x \text{ contained in } U \cap V.$$

The structure of a ring in $\mathcal{O}_{X,x}$ is induced by the ring structure of any $\mathcal{O}(U)$. We shall call the equivalence class of $f \in \mathcal{O}(U)$, the *germ* of f at x. The maximal ideal $m_{X,x}$ of X at x is the ideal of germs of functions $f \in \mathcal{O}(U)$ which vanish at x.

Lemma 5. *(i) $m_{X,x}$ is the unique maximal ideal of $\mathcal{O}_{X,x}$.*

(ii) If X is affine, the canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ induces an isomorphism $\mathcal{O}(X)_{m_x} \cong \mathcal{O}_{X,x}$.

(iii) If X is affine, the canonical homomorphism $\mathcal{O}(X) \to \mathcal{O}_{X,x}$ induces an isomorphism of fields $k = \mathcal{O}(X)/m_x \to \mathcal{O}_{X,x}/m_{X,x}$.

Proof. (i) It suffices to show that every element $f \in \mathcal{O}_{X,x} \setminus m_{X,x}$ is invertible. Assume that f is regular on an open affine set U containing x. Since $f(x) \neq 0, x$ is contained in the open principal affine subset $V = D(f)$ of U. Hence $1/f \in \mathcal{O}(V)$ is invertible on V. So $1/f$ is invertible in $\mathcal{O}_{X,x}$.

(ii) For any $\phi \in \mathcal{O}(X) \setminus m_x$ its germ in $\mathcal{O}_{X,x}$ is invertible. By the universal property of localizations, this defines a homomorphism $\mathcal{O}(X)_{m_x} \to \mathcal{O}_{X,x}$. It is obviously injective. Let $f_x \in \mathcal{O}_{X,x}$ be the germ of a function $f \in \mathcal{O}(U)$, where U is an open affine neighborhood of x. Replacing U by a principal open subset $D(\phi) \subset U$, we may assume that $U = D(\phi)$ and $f = F/\phi^n$, where $F \in \mathcal{O}(X)$. Since $\phi(x) \neq 0$, we get that ϕ does not belong to m_x, and hence $f \in \mathcal{O}(X)_{m_x}$ and its germ at x equals f_x. This proves the surjectivity.

(iii) For any $U \subset V$, the restriction homomorphism $\mathcal{O}(V) \to \mathcal{O}(U)$ is identical on k and the image of the maximal ideal $m(V)_x$ (of functions on V vanishing at x) generates the maximal ideal $m(U)_x$ of $\mathcal{O}_{U,x}$. From this the assertion easily follows.

Proposition 2. *Let X be any algebraic k-set and $x \in X$. Then*

$$T(X)_x \cong Hom_k(m_{X,x}/m_{X,x}^2, K).$$

Proof. Let U be an affine open neighborhood of x. By definition the canonical map $\mathcal{O}(U) \to \mathcal{O}_{X,x}$ maps $m(U)_x$ into $m_{X,x}$. This defines a linear map

$$r(U) : m(U)_x/m(U)_x^2 \to m_{X,x}/m_{X,x}^2.$$

Assume $r(U)(s) = 0, s \neq 0$. Let s be the coset of a function ϕ on U vanishing at x. Then there exists an open subset V of U containing x such that its restriction to V belongs to m_x^2. Then $(dj)_x(\phi_x) = 0$, where $j : V \hookrightarrow U$. This contradiction shows that $r(U)$ is injective. For each $s \in m_{X,x}/m_{X,x}^2$, we can find an open subset V in U such that s belongs to the image of $r(V)$. Since $(dj)_x$ is surjective, s belongs to the image of $r(U)$. Thus $r(U)$ is bijective. Now the assertion follows from Lemma 3.

Corollary. *Let $f : X \to Y$ be a rational map inducing an isomorphism from an open neighborhood of a point $x \in X$ to an open neighborhood of $f(x)$. Then $T(X)_x \cong T(Y)_{f(x)}$.*

Proof. Indeed, f induces an isomorphism of local rings $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,f(x)}$.

We leave to the reader to define the differential $(df)_x : T(X)_x \to T(Y)_{f(y)}$ for a regular map of any algebraic sets and check the analog of Proposition 1.

Let X be an algebraic set and $T(X)_x$ be the Zariski tangent space at its point $x \in X(K)$. It is a vector space over K of finite dimension. In fact, since $m_{X,x}$ is finitely generated, the linear space $m_{X,x}/m_{X,x}^2$ is finite-dimensional. We have

$$\dim_K T(X)_x = \dim_K(m_{X,x}/m_{X,x}^2)^* = \dim_K(m_{X,x}/m_{X,x}^2).$$

We denote $\dim_K T(X)_x$ by $\mathrm{embdim}_x X$ and call it the *embedding dimension* of X at x. If x is contained in an affine open subset U which is isomorphic to a closed subset of some $\mathbf{A}^n(K)$, then $T(X)_x$ is a subspace of $T(\mathbf{A}^n(K))_x \cong K^n$ and

$$\dim_K T(X)_x \leq n.$$

This shows that X cannot be given by equations in less than $d = \mathrm{embdim}_x X$ unknowns in a neighborhood of x. Or, in other words, none of affine open neigbourhoods of x is isomorphic to a closed subset of $\mathbf{A}^n(K)$ with $n \leq d$.

Example 3. Let X be the union of the three coordinate axes in $\mathbf{A}^3(K)$. It is given by the system

$$Z_1Z_2 = Z_1Z_3 = Z_2Z_3 = 0.$$

The tangent space at the origin $x = (0,0,0)$ is the whole tangent space $T(\mathbf{A}^3(K))_x \cong K^3$. Thus $\mathrm{embdim}_x X = 3$. This shows that X is not isomorphic to the union of three lines in $\mathbf{A}^2(K)$.

Let us now show that $\mathrm{embdim}_x X \geq \dim X$ for any irreducible algebraic set X and the equality takes place for almost all points x (i.e., for all points belonging to a Zariski open subset of X). For this, we may obviously assume that X is affine.

Let $V = X(K)$ for some affine variety defined by a radical ideal in $k[Z_1, \ldots, Z_n]$. The set $T(V) = X(K[\varepsilon])$ is a subset of $K[\varepsilon]^n$ which can be thought as the vector space K^{2n}. It is easy to see that $T(X)$ is a closed algebraic subset of K^{2n} and the map $p = X(\phi) : T(X) \to X$, is a regular map (check it !). Note that the fibre $p^{-1}(x)$ is equal to the tangent space $T(X)_x$. Applying the theorem about the dimension of fibres of a regular map, we obtain

Proposition 3. *There exists a number d such that*

$$\dim_K T(X)_x \geq d$$

and the equality takes place for all points x belonging to an open subset of X.

We will show that the number d from above is equal to $\dim X$.

Lemma 6. *Let K be an algebraically closed field of characteristic p. Let $F \in K[Z_1, \ldots, Z_n]$ with all the partial derivatives $\partial F/\partial Z_i$ equal to zero. If $p = 0$, then F is a constant polynomial. If $p > 0$, then $F = G^p$ for some polynomial G.*

Proof. Write $F = \sum_r a_r Z^r$. Then

$$\frac{\partial F}{\partial Z_i} = \sum_r a_r (r \bullet e_i) Z^{r - e_i}$$

where $\bullet$ denotes the dot product of vectors and e_i is the i-th unit vector. If this polynomial is equal to zero, then $a_r(r \bullet e_i) = 0$ for all r. Assume that $a_r \neq 0$. If $\mathrm{char}(k) = 0$ this implies that $r \bullet e_i = 0$. In particular, if all $\frac{\partial F}{\partial Z_i} = 0$, we get $r = 0$, i.e., F is a constant polynomial. If $\mathrm{char}(k) = p > 0$, we obtain that p divides $r \bullet e_i$, i.e., $r = pr'$ for some vector r'. Thus

$$F = \sum_r a_r Z^r = \sum_r a_r (Z^{r'})^p = (\sum_r a_r^{1/p} Z^{r'})^p = G^p,$$

where $G = \sum_r a_r^{1/p} Z^{r'}$.

Theorem 1. *Let X be an irreducible algebraic set and $d = \min\{embdim_x X, x \in X\}$. Then*

$$d = \dim X.$$

Proof. Obviously, it suffices to find an open subset U of X where $\mathrm{embdim}_x X = \dim X$ for all $x \in U$. Replacing X by an open affine set, we may assume that X is isomorphic to an open subset of a hypersurface $V(F) \subset \mathbf{A}^n(K)$ for some irreducible polynomial F (Theorem 2 of Lecture 4). This shows that we may assume that $X = V(F)$. For any $x \in X$, the tangent space $T(X)_x$ is given by one equation

$$\frac{\partial F}{\partial Z_1}(x)b_1 + \ldots + \frac{\partial F}{\partial Z_n}(x)b_n = 0.$$

Clearly, its dimension is equal to $n - 1 = \dim X$ unless all the coefficients are zeroes. The set of common zeroes of the polynomials $\frac{\partial F}{\partial Z_i}$ is a closed subset of $\mathbf{A}^n(K)$ contained in each hypersurface $V(\frac{\partial F}{\partial Z_i})$. Obviously $\frac{\partial F}{\partial Z_i} \notin (F)$ unless it is equal to zero (compare the degrees). Now the assertion follows from Lemma 6.

Obviously, the assertion of the previous theorem is not true for a reducible set. To see this it is sufficient to consider the union of two sets of different dimension. It is easy to modify the statement to extend it to the case of reducible sets.

Definition. The dimension of X at a point x is the maximum $\dim_x X$ of the dimensions of irreducible components of X containing x.

Corollary. *Let X be an algebraic set and $x \in X$. Then*

$$\mathrm{embdim}_x X \geq \dim_x X.$$

Proof. Let Y be an irreducible component of X containing x. Obviously $T(Y)_x \subset T(X)_x$. Hence

$$\dim_x Y \leq \mathrm{embdim}_x Y \leq \mathrm{embdim}_x X.$$

This proves the assertion.

Definition. A point x of an algebraic set X is said to be *nonsingular* (or *simple*, or *smooth*) if $\mathrm{embdim}_x X = \dim_x X$. Otherwise, it is said to be *singular*. An algebraic set X is said to be nonsingular (or smooth) if all its points are nonsingular. Otherwise X is said to be singular.

We already know how to recognize whether a point is nonsingular.

Theorem 2. *(The Jacobian criterion of a nonsingular point). Assume that X is an affine algebraic k-set given by a system of equations $F_1(Z) = \ldots = F_r(Z) = 0$ in $\mathbf{A}^n(K)$. Then $x \in X$ is nonsingular if and only if rk $J(x) = n - \dim_x X$, where*

$$J(x) = \begin{pmatrix} \frac{\partial F_1}{\partial Z_1}(x) & \ldots & \frac{\partial F_1}{\partial Z_n}(x) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \frac{\partial F_r}{\partial Z_1}(x) & \ldots & \frac{\partial F_r}{\partial Z_n}(x) \end{pmatrix}.$$

Problems.

1. Assume $k = K$ is algebraically closed field of characteristic 0. Show that, up to a projective automorphism of $\mathbf{P}^2(K)$, there are only two irreducible singular plane cubic curves.
2. Prove that $T(X \times Y)(x, y) \cong T(X)_x \oplus T(Y)_y$. Using this show that if x is a nonsingular point of X and y is a nonsingular point of Y, then (x, y) is a nonsingular point of $X \times Y$.
3. Let X be a closed subset of $\mathbf{A}^n(K), x = (a_1, \dots, a_n) \in X$ and $f : \mathbf{A}^1(K) \to \mathbf{A}^n(K)$ given by $t \mapsto (b_1 t + a_1, \dots, b_n t + a_n)$. Let (Z^r) be the ideal of $\mathcal{O}(\mathbf{A}^1(K)) \cong k[Z]$ generated by the functions $f^*(\phi), \phi \in I(X)$. Show that $(a_1 + b_1\varepsilon, \dots, a_n + b_n\varepsilon) \in K[\varepsilon]^n$ is a tangent vector of X if and only if $r > 1$. Note that r can be interpreted as the intersection multiplicity of X and the line $f(\mathbf{A}^1(K))$ at x.
4. Suppose a hypersurface $X = V(F)$ of degree > 1 in $\mathbf{P}^n(K)$ contains a linear subspace E of dimension $r \geq n/2$. Show that X has singular points contained in E.
5. Find singular points of the Steiner quartic $V(T_0^2T_1^2 + T_1^2T_2^2 + T_0^2T_2^2 - T_0T_1T_2T_3)$ in $\mathbf{P}^3(K)$.
6. Let X be a surface in $\mathbf{P}^3(K)$. Assume that X contains three nocoplanar lines passing through a point $x \in X$. Show that this point is singular.

7*. By analyzing the proof of Theorem 3 from Lecture 12, show that any nonsingular cubic surface contains exactly 27 lines.

8*. Let $SL_k(n)$ be the subfunctor of $GL_k(n)$ that assigns to a k-algebra K the group of unimodular matrices (i.e., of determinant 1). Show that $SL_k(n)$ is an affine algebraic k-variety. Compute the tangent space of both $SL_k(n)$ and $GL_k(n)$ at some point $A \in SL_k(n)(K) \subset GL_k(n)(K)$. Compare the answer with the definition of the Lie algebra gl_n and sl_n.

9*. Let $G_k(r+1, n+1)$ be the Grassmann variety over k. For every $M \in G_k(r+1, n+1)(K)$ show that the tangent space of $G_k(r+1, n+1)$ at M is naturally identified with $Hom_K(M, K^{n+1}/M)$.

Lecture 14. LOCAL PARAMETERS

In this lecture we will give some other properties of nonsingular points. First, a definition:

Definition. Let X be an algebraic k-set and $x \in X$. The elements $f_1, \ldots, f_n \in m_{X,x}$ are said to be *local parameters* at x if their residues mod $m_{X,x}^2$ form a basis of the linear space $m_{X,x}/m_{X,x}^2$ over the field $k(x) = \mathcal{O}_{X,x}/m_{X,x}$.

Let us show that a set of local parameters is a generating set for the maximal ideal $m_{X,x}$. In particular, if x is a nonsingular point of an algebraic set X, then $m_{X,x}$ can be generated by $n = \dim_x X$ elements. Clearly, the residues mod $m_{X,x}^2$ of every set of generators of $m_{X,x}$ span $m_{X,x}/m_{X,x}^2$. Thus the minimal number of generators of $m_{X,x}$ does not exceed n and it is equal to n exactly when x is a nonsingular point.

Lemma (of Nakayama). *Let A be a local ring with maximal ideal m, and let M be a finitely generated A-module. Assume that $M = N + mM$ for some submodule N of M. Then $M = N$.*

Proof. Replacing M by the factor module M/N, we may assume that $N = 0$. Let $f_1, \ldots, f_r$ be a set of generators of M. Since $mM = M$, we may write

$$f_i = \sum_{j=1}^{r} a_{ij} f_j, \quad i = 1, \ldots, r,$$

for some a_{ij} in m. Let $R = (a_{ij})$ be the matrix of coefficients. Since $(f_1, \ldots, f_r)$ is a solution of the homogeneous system of equations $R \cdot x = 0$, by Cramer's rule,

$$\det(R) f_i = 0, i = 1, \ldots, r.$$

However, $\det(R) = (-1)^r + a$ for some $a \in m$ (being the value of the characteristic polynomial of R at 1). In particular $\det(R)$ is invertible in A. This implies that $f_i = 0$ for all i, i.e., $M = \{0\}$.

Corollary. *Let A be a local Noetherian ring and m be its maximal ideal. Elements $a_1, \ldots, a_r$ generate m if and only if their residues modulo m^2 span m/m^2 as a vector space over $k = A/m$. In particular, the minimal number of generators of the maximal ideal m is equal to the dimension of the vector space m/m^2.*

Proof. Let $M = m, N = (a_1, \ldots, a_r)$. Since A is Noetherian, M is a finitely generated A-module and N its submodule. By the assumption, $M = mM + N$. By the Nakayama lemma, $M = N$.

Proposition 1. *Let X be an algebraic k-set, $x \in X$ and $f_1, \ldots, f_r \in m_{X,x}$. The following properties are equivalent:*
(i) $f_1, \ldots, f_r$ is a set of local parameters;
(ii) $f_1, \ldots, f_r$ generate $m_{X,x}$ and $r = \mathrm{embdim}_x X$;
(iii) $f_1, \ldots, f_r$ is a minimal set of generators of $m_{X,x}$.

Proof. $(i) \iff (ii)$. By definition $r = \dim_{k(x)} m_{X,x}/m_{X,x}^2$. It follows from the previous corollary that $f_1, \ldots, f_r$ generate $m_{X,x}$.

$(ii) \iff (iii)$ If $\{f_1, \ldots, f_r\}$ contains a proper subset of generators of $m_{X,x}$, then the cosets of the elements of this subset span the space $m_{X,x}/m_{X,x}^2$. Thus its dimension is less than $r = \dim_{k(x)} m_{X,x}/m_{X,x}^2$. This is absurd.

$(iii) \iff (i)$. The residues $(f_i)_x$ of the f_i modulo $m_{X,x}^2$ span $m_{X,x}/m_{X,x}^2$. It suffices to show that they are linearly dependent. Assume that it is not true. Then a proper subset of $\{(f_1)_x, \ldots, (f_r)_x\}$ spans $m_{X,x}/m_{X,x}^2$. By the previous corollary the representatives of the elements $(f_i)_x$ generate $m_{X,x}$. This contradicts the minimality of the set $\{f_1, \ldots, f_r\}$.

Definition. A Noetherian local ring A is called *regular* if its maximal ideal m is generated by $n = \dim A$ elements.

Corollary. *Let x be a point of an algebraic set X and $n = \dim_x X$. The following properties are equivalent:*
(i) x is a nonsingular point;
(ii) any set of local parameters at x consists of n elements;
(iii) the minimal number of generators of $m_{X,x}$ is equal to n;
(iv) the local ring $\mathcal{O}_{X,x}$ is regular.

Examples. 1. Let X be given by the equation $Y^2 + X^3 = 0$ and $x = (0,0)$. The maximal ideal $m_{X,x}$ is generated by the residues of the two unknowns. It is easy to see that this ideal is not principal. The reason is clear: x is a singular point of X and $\mathrm{embdim}_x X = 2 > \dim_x X = 1$. On the other hand, if we replace X by the set given by the equation $Y^2 + X^3 + X = 0$, then $m_{X,x}$ is principal. It is generated by the germ of the function Y. Indeed, $Y^2 = -X(X^2+1)$ and the germ of X^2+1 at the origin is obviously invertible. Note that the maximal ideal $m(X)_x$ of $\mathcal{O}(X)$ is not principal.
2. Let $x = (a_1, \ldots, a_n) \in X = \mathbf{A}^n(K)$, the germs of the polynomials $Z_i - a_i, i = 1, \ldots, n$, form a system of parameters at the point x. For any polynomial $F(Z_1, \ldots, Z_n)$ we can write

$$F(Z_1, \ldots, Z_n) = F(x) + \sum_{i=1}^{n} \frac{\partial F}{\partial Z_i}(x)(Z_i - a_i) + G(Z_1, \ldots, Z_n),$$

where $G(Z_1, \ldots, Z_n) \in m_x^2$. Thus the cosets dZ_i of $Z_i - a_i \bmod m_{X,x}^2$ form a basis of the linear space $m_{X,x}/m_{X,x}^2$ and the germ $F_x - F(x) = F(Z_1, \ldots, Z_n) - F(x) \bmod m_{X,x}^2$ is a linear combination of $dZ_1, \ldots, dZ_n$ with the coefficients equal to the partial derivatives evaluated at x. Let $\frac{\partial}{\partial Z_i}$ denote the basis of $T(X)_x$ dual to the basis $dZ_1, \ldots, dZ_n$. Then the value of the tangent vector $\sum_i \alpha_i \frac{\partial}{\partial Z_i}$ at $F_x - F(x)$ is equal to

$$\sum_{i=1}^{n} \alpha_i \frac{\partial F}{\partial Z_i}(x).$$

This is also the value at F of the derivation of $k[Z_1, \ldots, Z_n]$ defined by the tangent vector $\sum_i \alpha_i \frac{\partial}{\partial Z_i}$.

Let $f : X = \mathbf{A}^n(K) \to Y = \mathbf{A}^m(K)$ be a regular map given by a homomorphism

$$k[T_1, \ldots, T_m] \to k[Z_1, \ldots, Z_n], T_i \to P_i(Z_1, \ldots, Z_n).$$

Let $\partial_x = \sum_i \alpha_i \frac{\partial}{\partial Z_i} \in T(X)_x$, then

$$(df)_x(\partial_x)(T_i) = \partial_x(f^*(T_i)) = \partial_x(P_i(Z_1, \ldots, Z_n)) =$$

$$= \sum_{j=1}^{n} \alpha_j \frac{\partial P_i}{\partial Z_j}(x) = \sum_{k=1}^{m} \sum_{j=1}^{n} \alpha_j \frac{\partial P_i}{\partial Z_j}(x) \frac{\partial}{\partial T_k}(T_i).$$

From this we infer that the matrix of the differential $(df)_x$ with respect to the bases $\frac{\partial}{\partial Z_1}, \ldots, \frac{\partial}{\partial Z_n}$ and $\frac{\partial}{\partial T_1}, \ldots, \frac{\partial}{\partial T_m}$ of $T(X)_x$ and $T(Y)_{f(x)}$, respectively, is equal to

$$\begin{pmatrix} \frac{\partial P_1}{\partial Z_1} & \cdots & \frac{\partial P_1}{\partial Z_n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \frac{\partial P_m}{\partial Z_1} & \cdots & \frac{\partial P_m}{\partial Z_n} \end{pmatrix}.$$

Lemma 2. *Let $\bar{f}_1, \ldots, \bar{f}_r \in m_{X,x}$ be represented by regular functions $f_1, \ldots, f_r$ on an open affine neighborhood U of x and let $Y = V((f_1, \ldots, f_r)) \subset U$. Let E be the subspace of $m_{X,x}/m_{X,x}^2$ generated by the cosets of all elements from rad $(f_1, \ldots, f_r)$. Then*

$$T(Y)_x = \{\phi \in (m_{X,x}/m_{X,x}^2)^* = T(X)_x : \phi(E) = 0\}.$$

Proof. Let $p : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,x}$ be the surjective homomorphism corresponding to the restriction of regular functions to a closed subset. It is clear that $p^{-1}(m_{Y,x}^r) = (m_{X,x})^r$ and $\mathrm{Ker}(p) = \mathrm{Rad}(f_1, \ldots, f_r) = I$. The homomorphism p induces a linear map: $p_x : m_{X,x}/m_{X,x}^2 \to m_{Y,x}/m_{Y,x}^2$ whose kernel is the subspace generated by the residues f_x of f mod $m_{X,x}^2, f \in I$. Every linear function on $m_{Y,x}/m_{Y,x}^2$ composed with the map p_x defines a linear function on $m_{X,x}/m_{X,x}^2$ vanishing at every $(f_i)_x$. Conversely, if ϕ is such a linear function on $m_{X,x}/m_{X,x}^2$, then it vanishes at each $(f_i)_x$.

Definition. Let $Y_1, \ldots, Y_r$ be closed subsets of an algebraic set $X, x \in Y_1 \cap \ldots \cap Y_r$. We say that $Y_1, \ldots, Y_r$ intersect transversally at the point x if X is nonsingular at x and

$$\mathrm{codim}(T(\bigcap_{i=1}^{r} Y_i)_x, T(X)_x) = \sum_{i=1}^{r} \mathrm{codim}_x(Y_i, X).$$

Proposition 2. *Let $Y_1, \ldots, Y_r$ be closed subsets of an algebraic set X which intersect transversally at $x \in X$. Then*

(i) $T(Y_1)_x, \ldots, T(Y_r)_x$ intersect transversally in $T(X)_x$ (i.e., the sum $\sum_i T(Y_i)_x$ is equal to the direct sum $\oplus_i T(Y_i)_x$) and $\cap_i T(Y_i)_x = T(\cap_i Y_i)_x$;

(ii) x is a nonsingular point of $Y_1 \cap \ldots \cap Y_r$.

(iii) each Y_i is nonsingular at x.

(iv) for any $\{i_1,\ldots,i_k\}\subset\{1,\ldots,r\}, Y_{i_1},\ldots,Y_{i_r}$ intersect transversally at x.

Proof. We know that $\text{codim}(T(Y_i)_x, T(X)_x)\le \text{codim}_x(Y_i,X)$ for any $i\le r$. Hence

$$\sum_{i=1}^r \text{codim}_x(Y_i,X)=\text{codim}(\cap_{i=1}^r T(Y_i)_x, T(X)_x)\le\sum_{i=1}^r \text{codim}(T(Y_i)_x,T(X)_x)\le\sum_{i=1}^r\text{codim}_x(Y_i,X)$$

which shows that all the inequalities must be equalities. This yields

$$\text{codim}(\bigcap_i T(Y_i)_x, T(X)_x)=\sum_{i=1}^r\text{codim}(T(Y_i)_x,T(X)_x),$$

showing that the linear subspaces $T(Y_i)_x$ of $T(X)_x$ intersect transversally, and

$$\text{codim}(T(Y_i)_x,T(X)_x)=\text{codim}_x(Y_i,X),\quad i=1,\ldots,r,$$

i.e., x is a nonsingular point of Y_i. Obviously, $T(Y_1\cap\ldots\cap Y_r)_x\subset T(Y_1)_x\cap\ldots\cap T(Y_r)_x$. Thus

$$\sum_{i=1}^r\text{codim}(Y_i,X)=\text{codim}(\bigcap_{i=1}^r T(Y_i)_x,T(X)_x)\le\text{codim}(T(Y_1\cap\ldots\cap Y_r),T(X)_x)$$

$$\le\text{codim}_x(\bigcap_{i=1}^r Y_i,X)\le\sum_{i=1}^r\text{codim}_x(Y_i,X).$$

This gives

$$T(Y_1\cap\ldots\cap Y_r)_x=T(Y_1)_x\cap\ldots\cap T(Y_r)_x$$

and

$$\text{dim}T(Y_1\cap\ldots\cap Y_r)_x=\text{dim}_x(\cap_i Y_i),$$

proving that x is a nonsingular point of $\cap_i Y_i$. This checks the assertion (i) - (iii).

To prove (iv), we may assume that $\{i_1,\ldots,i_j\}=\{1,\ldots,j\}$. We know that $T(Y_1)_x,\ldots,T(Y_j)_x$ intersect transversally in $T(X)_x$ and x is a nonsingular point of Y_i. This gives us

$$\text{codim}(\bigcap_{i\le j}T(Y_i)_x,T(X)_x)=\sum_{i=1}^r\text{codim}(T(Y_i)_x,T(X)_x)=\sum_{i=1}^r\text{codim}_x(Y_i,X)$$

which proves (iv).

Next we will show that every function from $\mathcal{O}_{X,x}$ can be expanded into a formal power series in a set of local parameters at x.

Recall that the k-algebra of formal power series in n variables $k[[Z]]=k[[Z_1,\ldots,Z_n]]$ consists of all formal (infinite) expressions

$$P=\sum_{\mathbf{r}}a_{\mathbf{r}}Z^{\mathbf{r}},$$

where $\mathbf{r}=(r_1,\ldots,r_n)\in\mathbf{N}^n, a_{\mathbf{r}}\in k, Z^{\mathbf{r}}=Z_1^{r_1}\ldots Z_n^{r_n}$. The rules of addition and multiplicaton are defined naturally (as for polynomals). Equivalently, $k[[Z]]$ is the set of functions $P:\mathbf{N}^n\to k,\mathbf{r}\to a_{\mathbf{r}}$, with the usual addition operation and the operation of multiplication defined by the convolution of functions:

$$(P*Q)(\mathbf{r})=\sum_{\mathbf{i}+\mathbf{j}=\mathbf{r}}P(\mathbf{i})Q(\mathbf{j}).$$

The polynomial k-algebra $k[Z_1,\dots,Z_n]$ can be considered as a subalgebra of $k[[Z_1,\dots,Z_n]]$. It consists of functions with finite support. Clearly every formal power series $P \in k[[Z]]$ can be written as a formal sum $P = \sum_j P_j$, where $P_j \in k[Z_1,\dots,Z_n]_j$ is a homogeneous polynomal of degree j.

We set

$$P_{[r]} = P_0 + P_1 + \ldots + P_r.$$

This is called the *r-truncation* of P.

Theorem 1 (Taylor expansion). *Let x be a nonsingular geometric point of an algebraic set X of dimension n, and $\{f_1,\dots,f_n\}$ be a set of local parameters at x. There exists a unique injective homomorphism $\phi : \mathcal{O}_{X,x} \hookrightarrow K[[Z_1,\dots,Z_n]]$ such that for every $i \geq 0$*

$$f - \phi(f)_{[i]}(f_1,\dots,f_n) \in m_{X,x}^{i+1}.$$

Proof. Take any $f \in \mathcal{O}_{X,x}$, we denote by $f(x)$ the image of f in $K = \mathcal{O}_{X,x}/m_{X,x}$ then $f - f(x) \in m_{X,x}$. Since the local parameters $f_1,\dots,f_n$ generate $m_{X,x}$, we can find elements $g_1,\dots,g_n \in \mathcal{O}_{X,x}$ such that

$$f = f(x) + g_1 f_1 + \ldots + g_n f_n.$$

Replacing f by g_i, we can write similar expressions for the $g_i's$. Plugging them into the above expresson for f, we obtain

$$f = f(x) + \sum_i g_i(x) f_i + \sum_{ij} h_{ij} f_i f_j,$$

where $h_{ij} \in \mathcal{O}_{X,x}$. Continuing in this way, we will find a formal power series $P = \sum_j P_j$ such that

$$(*) \qquad f - P_{[r]}(f_1,\dots,f_n) \in m_{X,x}^{r+1} \quad \text{for any } r \geq 0.$$

Let us show that $f \mapsto P$ defines an injective homomorphism $\mathcal{O}_{X,x} \to k[[Z]]$ satisfying the assertion of the theorem. First of all, we have to verify that this map is well defined, i.e. property $(*)$ determines P uniquely. Suppose there exists another formal power series $Q(Z) = \sum_j Q_j$ such that

$$f - Q_{[r]}(f_1,\dots,f_n) \in m_{X,x}^{r+1} \quad \text{for any } r \geq 0.$$

Let $r = \min\{j : Q_j \neq P_j\}$ and $F = Q_j - P_j \in k[Z_1,\dots,Z_n]_r \setminus \{0\}$. Taking into account $(*)$, we obtain that $F(f_1,\dots,f_n) \in m_{X,x}^{r+1}$. Making an invertible change of variables, we may assume that $F(0,\dots,0,1) \neq 0$, i.e.,

$$F(f_1,\dots,f_n) = G_0 f_n^r + G_1(f_1,\dots,f_{n-1}) f_n^{r-1} + \ldots + G_r(f_1,\dots,f_{n-1})$$

where $G_i(Z_1,\dots,Z_{n-1}) \in k[Z_1,\dots,Z_{n-1}]_i, G_0 \neq 0$. Since $f_1,\dots,f_n$ generate $m_{X,x}$, we can write

$$F(f_1,\dots,f_n) = H_1(f_1,\dots,f_n) f_n^r + H_2(f_1,\dots,f_{n-1}) f_n^{r-1} + \ldots + H_{r+1}(f_1,\dots,f_{n-1}),$$

where $H_i \in k[Z_1,\dots,Z_{n-1}]_i$. After subtracting the two expressions, we get

$$(G_0 - H_1(f_1,\dots,f_n)) f_n^r \in (f_1,\dots,f_{n-1}).$$

Since $H_1(f_1,\dots,f_n)\in m_{X,x}, G_0 - H_1(f_1,\dots,f_n)$ is invertible and $f_n^r \in (f_1,\dots,f_{n-1})$. Passing to the germs, we find that $m_{X,x} = (f_1,\dots,f_n) \subset \mathrm{rad}(f_1,\dots,f_{n-1})$, and hence $(f_1,\dots,f_n) = (f_1,\dots,f_{n-1})$ because $m_{X,x}$ is a maximal ideal. But this contradicts Proposition 1.

We leave to the reader to verify that the constructed map $\phi : \mathcal{O}_{X,x} \to k[[Z]]$ is a ring homomorphism. Let us check now that it is injective. It follows from the definition of this map that $\phi(f) = 0$ implies $f \in (m_{X,x})^r$ for all $r \geq 0$. Let J be the homogeneous ideal in $k[Z_1,\dots,Z_n]$ generated by all homogeneous polynomals F such that $F(f_1,\dots,f_n) = f$. Let $F_1,\dots,F_j$ be a finite subset of such polynomials which generate this ideal, and let $N = \max\{\deg F_i, i = 1,\dots,j\}$. Since $f \in m_{X,x}^{N+1}$ there exists $F \in k[Z]_{N+1}$ such that $F = \sum_i G_i F_i$ for some homogeneous $G_i \in k[Z]$ of positive degree and

$$f = F(f_1,\dots,f_n) = \sum_i G_i(f_1,\dots,f_n)F_i(f_1,\dots,f_n) = \sum_i G_i(f_1,\dots,f_n)f.$$

Thus,

$$f(1 - \sum_i a_i) = 0$$

where $a_i = G_i(f_1,\dots,f_n) \in m_{X,x}$. Since $(1 - \sum_i a_i)$ is invertible, f must be zero.

Definition. Let $\phi : \mathcal{O}_{X,x} \to k[[Z_1,\dots,Z_n]]$ be the injective homomorphism constructed in Theorem 1. The image $\phi(f)$ of an element $f \in \mathcal{O}_{X,x}$ is called the Taylor expansion of f at x with respect to the local parameters $f_1,\dots,f_n$.

Corollary 1. *The local ring $\mathcal{O}_{X,x}$ of a nonsingular point does not have zero divisors.*

Proof. $\mathcal{O}_{X,x}$ is isomorphic to a subring of the ring $k[[Z]]$ which, as is easy to see, does not have zero divisors .

Corollary 2. *A nonsingular point of an algebraic set X is contained in a unique irreducible component of X.*

Proof. This immediately follows from Corollary 1. Indeed, assume $x \in Y_1 \cap Y_2$ where Y_1 and Y_2 are irreducible components of X containing the point x. Replacing X by a small open affine neighborhood, we may find a regular function f_1 vanishing on Y_1 but not vanishing on the whole Y_2. Similarly, we can find a function f_2 vanishing on $X \setminus Y_1$ and not vanishing on the whole Y_1. The product $f = f_1 f_2$ vanishes on the whole X. Thus the germs of f_1 and f_2 are the zero divisors in $\mathcal{O}_{X,x}$. This contradicts the previous corollary.

Remarks. 1. Note the analogy with the usual Taylor expansion which we learn in Calculus. The local parameters are analogous to the differences $\Delta x_i = x_i - a_i$. The condition $f - [P]_r(f_1,\dots,f_n) \in m_{X,x}^{r+1}$ is the analog of the convergence: the difference between the function and its truncated Taylor expansion vanishes at the point $x = (a_1,\dots,a_n)$ with larger and larger order. The previous theorem shows that a regular function on a nonsingular algebraic set is like an analytic function: tits Taylor expansion converges to the function.
2. For every commutative ring A and it proper ideal I, one can define the I-adic formal completion of A as follows. Let $p_{n,k} : A/I^n \to A/I^k$ be the canonical homomorphism of factor rings ($n \geq k$). Set

$$\hat{A}_I = \{(\dots,a_k,\dots,a_n\dots) \in \prod_{r\geq 0}(A/I^r) : p_{n,k}(a_n) = a_k \quad \text{for all } n \geq k\}.$$

It is easy to see that $\hat{A}_I$ is a commutative ring with respect to the addition and multiplication defined coordinatewise. We have a canonical homomorphism:

$$i : A \to \hat{A}_I, a \mapsto (a_0, a_1, \ldots, a_n, \ldots)$$

where a_n = residue of a modulo I^n. Note the analogy with the ring of p-adic numbers which is nothing else as the formal completion of the local ring $\mathbf{Z}_{(p)}$ of rational numbers $a/b, p \not| b$.

If A is a local ring with maximal ideal m, then $\hat{A}_m$ is denoted by $\hat{A}$ and is called the *formal completion* of A. Two local rings are called *formally isomorphic* if their formal completions are isomorphic. One can interpret the theorem of this lecture by stating that the formal completion of the local ring at a nonsingular point of an algebraic set is isomorphic to the ring of formal power series.

Let (X, x) be a pair that consists of an algebraic set X and its point $x \in X$. Two such pairs are called *locally isomorphic* if the local rings $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are isomorphic. They are called *formally isomorphic* if the completions of the local rings are isomorphic. Thus any pair (X, x) where x is a nonsingular point of X is locally formally isomorphic to a pair $(\mathbf{A}^n(K), 0)$ where $n = \dim_x X$. Compare this with the definion of a smooth (or complex manifold).

Theorem 2. *A regular local ring is a UFD (= factorial ring).*

The proof of this non-trivial result can be found in Zariski-Samuel's Commutative Agebra, vol. II. See the sketch of this proof in Shafarevich's book, Chapter II, §3. It uses an embedding of a regular ring into the ring of formal power series.

Corollary. *Let X be an algebraic set, $x \in X$ be its nonsingular point, and Y be a closed subset of codimension 1 which contains x. Then there exists an open subset U containing x such that $Y \cap U = V(f)$ for some regular function on U.*

Proof. Let V be an affine open set, $g \in I(Y \cap V)$, and let g_x be the germ of g at x and f_x be a prime factor of g_x which has a representative $f \in \mathcal{O}(U)$ vanishing on $Y \cap U$ for some smaller affine neighborhood U of x. At this point we may assume that $X = U$. Since $V(f) \supset Y$ and $\dim V(f) = \dim Y$, Y is equal to some irreducible component of $Y \cap U$, i.e., $V(f) = Y \cup Z$ for some closed subset of U. If $x \in Z$, then there exist regular functions h and h' on X such that $hh' \equiv 0$ on $V(f)$ but $h \not\equiv 0$ on Y and $h' \not\equiv 0$ on Z. By Hilbert's Nullstellensatz, $(hh')^r \in (f)$. Passing to the germs, we obtain that $f_x | (h_x h'_x)^r$. Since $\mathcal{O}_{X,x}$ is factorial, we obtain that $f_x | h_x$ or $f_x | h'_x$. Therefore for some open neighborhood $U' \subset U$, either $h|U'$ or $h'|U'$ vanishes at the point x. This contradicts the choice of h and h'. This shows that $x \notin Z$, and replacing U by a smaller open subset, the proof is complete.

Here is the promised application.

Recall that a rational map $f : X - \to Y$ from an irreducible algebraic set X to an algebraic set Y is a regular map of an open subset of X. Two rational maps are said to be equal if they coincide on an open subset of X. Replacing X and Y by open affine subsets, we find ourselves in the affine situation of Lecture 4. We say that a rational map $f : X - \to Y$ is defined at a point $x \in X$ if it can be represented by a regular map defined on an open subset containing the point x. A point x where f is not defined is called a *point of indeterminacy* of f.

Theorem 3. *Let $f : X - \to Y$ be a rational map of a nonsingular algebraic set X to a projective set Y. Then the set of indeterminacy points of f is a closed subset of X each irreducible component of which is of codimension ≥ 2.*

Proof. Since $Y \subset \mathbf{P}^n(k)$ for some n, we may assume that $Y = \mathbf{P}^n(K)$. Let U be the maximal open subset where f is represented by a regular map $f : U \to \mathbf{P}^n(K)$, and $Z = X \setminus U$. Assume Z contains an irreducible component of codimension 1. By Corollary to Theorem 2, for any $x \in Z$ there exists an open neighborhood V of x such that $Z \cap V = V(\phi)$ for some regular function ϕ on V. Restricting f to some smaller subset of $D(\phi) = V \setminus V(\phi)$ we may assume that $f|D(\phi)$ is given by $n+1$ regular functions $\phi_1, \ldots, \phi_{n+1}$ on $D(\phi)$. Since $\mathcal{O}_{X,x}$ is factorial, we may cancel the germs $(\phi_i)_x$ by their common divisor to assume that not all of them are divisible by the germ ϕ_x of ϕ. The resulting functions define the same map to $\mathbf{P}^n(K)$. It is not defined at the set of common zeroes of the functions ϕ_i. Its intersection with Z cannot contain any open neighborhood of x, hence is a proper closed subset of Z. This shows that we can extend f to a larger open subset contradicting the maximality of U.

Corollary. *Any rational map of a nonsingular curve to a projective set is a regular map. In particular, two nonsingular projective curves are birationally isomorphic if and only if they are isomorphic.*

This corollary is of fundamental importance. Together with a theorem on resolution of singularities of a projective curve it implies that the set of isomorphism classes of field extensions of k of transcedence degree 1 is in a bijective correspondence with the set of isomorphism classes of nonsingular projective algebraic curves over k.

Problems.

1. Using Nakayama's Lemma prove that a finitely generated projective module over a local ring is free.
2. A regular map $f : X \to Y$ is called *smooth* at a point $x \in X$ if $(df)_x : T(X)_x \to T(Y)_y$ is injective.
   (a) Assume $\mathrm{char}K = 0$. Show that the set of points $x \in X$ at which f is not smooth is a proper closed subset of X. Give a counterexample to this assertion if the characteristic is positive.
   (b) Let $f : \mathbf{A}^2(K) \to \mathbf{A}^2(K)$ be the map given by the formula $(a, b) \to (ab, a^2 + b^2)$. Find the set of points $x = (a, b)$ where f is not smooth.
3. Let A be a regular local ring of dimension d with maximal ideal m. Show that m^n/m^{n+1} is a vector space over $k = A/m$ of dimension $\binom{d+n-1}{n}$.
4. Let $X = V(F) \subset \mathbf{A}^2(K)$ where $F = Z_1^3 - Z_2(Z_2 + 1)$. Find the Taylor expansion at $(0, 0)$ of the function $Z_2 \mod (F)$ with respect to the local parameter $Z_1 \mod (F)$.
5. Give an example of a singular point $x \in X$ such that there exists an injective homomorphism $\mathcal{O}_{X,x} \to k[[Z_1]]$. Give an example of a curve X and a point $x \in X$ for which such homomorphism does not exist.
6. Let Y be a closed subset of X and $x \in Y$ be a nonsingular point of X. Prove that x is a nonsingular point of Y if and only if there exist local parameters $f_1, \ldots f_c, c = codim_x(Y, X)$ in $\mathcal{O}_{X,x}$ such that for some open affine neighborhood of x in X, the ideal of $Y \cap U$ is generated by representatives of $f_1, \ldots, f_c$.
7. Show that Theorem 3 is not true for singular projective algebraic curves.

8*. Let $X = V(Z_1Z_2 + F(Z_1, Z_2)) \subset \mathbf{A}^2(K)$ where F is a homogeneous polynomial of degree ≥ 3. Show that $\hat{\mathcal{O}}_{X,x} \cong K[[T_1, T_2]]/(T_1T_2)$.

Lecture 15. PROJECTIVE EMBEDDINGS

In this Lecture we shall address the following question: Given a projective algebraic k-set X, what is the minimal N such that X is isomorphic to a closed subset of $\mathbf{P}_k^N(K)$? We shall prove that $N \leq 2\dim X + 1$. For simplicity we shall assume here that $k = K$.

Definition. A regular map of projective algebraic sets $f : X \to \mathbf{P}^r(K)$ is called an *embedding* if it is equal to the composition of an isomorphism $f' : X \to Y$ and the identity map $i : Y \to \mathbf{P}^r(K)$, where Y is a closed subset of $\mathbf{P}^r(K)$.

Theorem 1. *A finite regular map $f : X \to Y$ of algebraic sets is an isomorphism if and only if it is bijective and for every point $x \in X$ the differential map $(df)_x : T(X)_x \to T(Y)_{f(x)}$ is injective.*

Proof. To show that f is an isomorphism it suffices to find an open affine covering of Y such that for any open affine subset V from this covering the homomorphism of rings $f^* : \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$ is an isomorphism. The inverse map will be defined by the maps of affine sets $V \to f^{-1}(V)$ corresponding to the inverse homomorphisms $(f^*)^{-1} : \mathcal{O}(f^{-1}(V)) \to \mathcal{O}(V)$. So we may assume that X and Y are affine and also irreducible.

Let $x \in X$ and $y = f(x)$. Since f is bijective, $f^{-1}(y) = \{x\}$. The homomorphism f^* induces the homomorphism of local rings $f_y^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Let us show that it makes $\mathcal{O}_{X,x}$ a finite $\mathcal{O}_{Y,y}$-module. Let $m \subset \mathcal{O}(Y)$ be the maximal ideal corresponding to the point y and let $S = \mathcal{O}(Y) \setminus m$. We know that $\mathcal{O}_{Y,y} = \mathcal{O}(Y)_S$, and, since finiteness is preserved under localizations, $\mathcal{O}(X)_{f^*(S)}$ is a finite $\mathcal{O}_{Y,y}$-module. I claim that $\mathcal{O}(X)_{f^*(S)} = \mathcal{O}_{X,x}$. Any element in $\mathcal{O}_{X,x}$ is represented by a fraction $\alpha/\beta \in Q(\mathcal{O}(X))$ where $\beta(x) \neq 0$. Since the map f is finite and bijective it induces a bijection from from the set $(V(\beta))$ of zeroes of β to the closed subset $f(V(\beta))$ of Y. Since $y \notin f(V(\beta))$ we can find a function $g \in S$ vanishing on $f(V(\beta))$. By Nullstellensatz, $f^*(g)^r = \beta\gamma$ for some $r > 0$ and some $\gamma \in \mathcal{O}(X)$. Therefore we can rewrite the fraction α/β in the form $\alpha\gamma/f^*(g)^r$ showing that it comes from $\mathcal{O}(X)_{f^*(S)}$. This proves the claim.

By assumption $f_y^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ induces a linear surjective map:

$$^t(df)_x : m_{Y,y}/m_{Y,y}^2 \to m_{X,x}/m_{X,x}^2$$

where "t" stands for the transpose map of the dual vector spaces. Let $h_1, \ldots, h_k$ be a set of local parameters of Y at the point y. Their images $f_y^*(h_1), \ldots, f_y^*(h_k)$ in $m_{X,x}$ span $m_{X,x}/m_{X,x}^2$. As follows from Lecture 14, this implies that $f_y^*(h_1), \ldots, f_y^*(h_k)$ generate $m_{X,x}$. Therefore,

$$f_y^*(m_{Y,y})\mathcal{O}_{X,x} = m_{X,x}.$$

Since $f_y^*(\mathcal{O}_{Y,y})$ contains constant functions, and $\mathcal{O}_{X,x} = K + m_{X,x}$, we get

$$\mathcal{O}_{X,x} = f_y^*(\mathcal{O}_{Y,y}) + m_{Y,y}\mathcal{O}_{X,x}.$$

Having proved that $\mathcal{O}_{X,x}$ is a finitely generated $\mathcal{O}_{Y,y}$-module we may apply Nakayama's lemma to obtain that

$$\mathcal{O}_{X,x} = f_y^*(\mathcal{O}_{Y,y}).$$

Therefore the map $f_y^* : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. It is obviously injective. Let $\phi_1, \ldots, \phi_m$ be generators of the $\mathcal{O}(Y)$-module $\mathcal{O}(X)$. The germs $(\phi_i)_x$ belong to $\mathcal{O}_{X,x} = f^*(\mathcal{O}_{Y,y})$ allowing us to write $(\phi_i)_x = f^*((\psi_i)_y)$, where ψ_i are regular functions on some affine open neighborhood V of $f(x)$. This shows that the germs of ϕ_i and $f^*(\psi_i)$ at the point x are equal. Hence, after replacing V by a smaller set V' if needed, we can assume that $\phi_i = f^*(\psi_i)$ for some open subset U of $f^{-1}(V)$. Since X is irreducible we can further assume that $U = f^{-1}(V)$. If we replace again V by a principal open subset $D(h) \subset Y$, we get $U = D(f^*(h)), \mathcal{O}(V) = \mathcal{O}(Y)_h, \mathcal{O}(U) = \mathcal{O}(X)_{f^*(h)}$, and the functions $\phi_i|U$ generate $\mathcal{O}(U)$ as a module over $\mathcal{O}(V)$. This implies that $f^* : \mathcal{O}(V) \to \mathcal{O}(f^{-1}(V))$ is surjective, hence an isomorphism. This proves the assertion.

Remark 1. The assumption of finiteness is essential. To see this let us take X to be the union of two disjoint copies of affine line with the origin in the second copy deleted, and let $Y = V(Z_1Z_2)$ be the union of two coordinate lines in $\mathbf{A}^2(K)$. We map the first copy isomorphically onto the lines $Z_1 = 0$ and map the second component of X isomorphically onto the line $Z_2 = 0$ with the origin deleted. It is easy to see that all the assumptions of Theorem 1 are satisfied except the finiteness. Obviously the map is not an isomorphism.

Definition. We say that a line ℓ in $\mathbf{P}^n(K)$ is *tangent* to an algebraic set X at a point $x \in X$ if $T(\ell)_x$ is contained in $T(X)_x$ (both are considered as linear subspaces of $T(\mathbf{P}^n(K))_x$).

Let E be a linear subspace in $\mathbf{P}^n(K)$ defined by a linear subspace $\bar{E}$ of K^{n+1}. For any point $x = (a_0, \ldots, a_n) \in E$ defined by the line $K(a_0, \ldots, a_n)$ in $\bar{E}$, the tangent space $T(E)_x$ can be identified with the factor space $\bar{E}/K(a_0, \ldots, a_n)$ (see Example 2 of Lecture 13). The inclusion $\bar{E} \subset K^{n+1}$ identifies it naturally with the subspace of $T(\mathbf{P}^n(K))_x = K^{n+1}/K(a_0, \ldots, a_n)$. Now let X be a projective subset of $\mathbf{P}^n(K)$ defined by a system of homogeneous equations $F_1(T_0, \ldots, T_n) = \ldots = F_m(T_0, \ldots, T_n) = 0$ and let $x \in X$. Then the tangent space $T(X)_x$ can be identified with the subspace of $T(\mathbf{P}^n(K))_x$ defined by the equations

$$(*) \qquad \sum_{j=0}^{n} \frac{\partial F_i}{\partial T_j}(x) b_j = 0, \quad i = 1, \ldots m.$$

This immediately follows from considering the set of solutions of the system $\{F_i = 0\}$ in $K[\epsilon]$. Note that for any homogeneous polynomial F of degree d we have the Euler formula:

$$dF = \sum_{j=0}^{n} \frac{\partial F_i}{\partial T_j} T_j.$$

This shows that the line $x = K(a_0, \ldots, a_n)$ is contained in the space of solutions of $(*)$ hence it makes sense to say that $(*)$ defines a subspace of $K^{n+1}/K(a_0, \ldots, a_n)$. Now we see that a line E is tangent to X at the point x if and only if $\bar{E}$ is contained in the space of solutions of $(*)$. In particular we obtain that the union of lines tangent to X is the linear subspace of $\mathbf{P}^n(K)$ defined by the system of linear homogeneous equations

$$(**) \qquad \sum_{j=0}^{n} \frac{\partial F_i}{\partial T_j}(x) T_j = 0, \quad i = 1, \ldots m.$$

Lemma 1. *Let X be a projective algebraic set in $\mathbf{P}^n(K), a \in \mathbf{P}^n(K) \subset X$, the linear projection map $p_a : X \to \mathbf{P}^{n-1}(K)$ is an embedding if and only if every line ℓ in $\mathbf{P}^n(K)$ passing through the point a intersects X in at most one point and is not tangent to X at any point.*

Proof. The induced map of projective sets $f : X \to Y = p_a(X)$ is finite and bijective. By Lemma 1, it suffices to show that the tangent map $(df)_x$ is injective. Without loss of generality we may assume that $a = (0, \dots, 0, 1)$ and the map p_a is given by restriction to X of the projection $p : \mathbf{P}^n(K) \setminus \{a\} \to \mathbf{P}^{n-1}(K)$ is given by the formula:

$$(T_0, \dots, T_n) \to (T_0, \dots, T_{n-1}).$$

For any point $x = (x_0, \dots, x_n) \neq a$, we can identify the tangent space $T(\mathbf{P}^n(K))_x$ with the quotient space $K^{n+1}/K(x_1, \dots, x_n)$, the tangent space $T(\mathbf{P}^{n-1}(K))_{p_a(x)}$ with $K^n/K(x_1, \dots, x_{n-1})$, and the differential $(dp_a)_x$ with the map $K^{n+1}/K(x_1, \dots, x_n) \to K^n/K(x_1, \dots, x_{n-1})$ induced by the projection $K^{n+1} \to K^n$. It is clear that its kernel is spanned by $Kx+K(0, \dots, 0, 1)/Kx$. But this is exactly the tangent space of the line ℓ spanned by the points $x = (x_0, \dots, x_n)$ and $a = (0, \dots, 0, 1)$. Thus the differential of the restriction of p_a to X is injective if and only if the tangent space of the line ℓ is not contained in the tangent space $T(X)_x$. This proves the assertion.

Lemma 2. *Let X be a quasi-projective algebraic subset of $\mathbf{P}^n(K)$ and $x \in X$ be its nonsingular point. Then the union $ET(X)_x$ of lines which are tangent to X at x is a projective subspace in $\mathbf{P}^n(K)$ of dimension equal to $d = \dim_x X$.*

Proof. We know that $ET(X)_x$ is the subspace of $\mathbf{P}^n(K)$ defined by the equations $(**)$. So it remains only to compute the dimension of this subspace. Since x is a nonsingular point of X, the dimension of $T(X)_x$ is equal to d. Now the result follows from comparing the equations $(*)$ and $(**)$. The first one defines the tangent space $T(X)_x$ and the second $ET(X)_x$. The (linear) dimension of solutions of both is equal to

$$d+1 = n+1 - rank(\frac{\partial F_i}{\partial T_j})(x) = dim_K T(X)_x + 1 = dim ET(X)_x + 1.$$

Note that the previous lemma shows that one can check whether a point of a projective set X is nonsingular by looking at the Jacobian matrix of homogeneous equations defining X.

We shall call the space $ET(X)_x$ the *embedded tangent space* of X at the point x.

Let

$$Z = \{(x, y, z) \in \mathbf{P}^n(K) \times \mathbf{P}^n(K) \times \mathbf{P}^n(K) : x, y, z \in \ell \quad \text{for some line } \ell\}.$$

This is a closed subset of $\mathbf{P}^n(K) \times \mathbf{P}^n(K) \times \mathbf{P}^n(K)$ defined by the equations expressing the condition that three lines $x = (x_0, \dots, x_n), y = (y_0, \dots, y_n), z = (z_0, \dots, z_n)$ are linearly dependent. The tri-homogeneous polynomials defining Z are the 3×3-minors of the matrix

$$\begin{pmatrix} T_0 & \dots & T_n \\ T'_0 & \dots & T'_n \\ T''_0 & \dots & T''_n \end{pmatrix}.$$

Let $p_{12} : Z \to \mathbf{P}^n(K) \times \mathbf{P}^n(K)$ be the projection map to the product of the first two factors. For any $(x, y) \in \mathbf{P}^n(K) \times \mathbf{P}^n(K)$

$$p_3(p_{12}^{-1}((x, y))) = \begin{cases} < x, y > & \text{if } x \neq y, \\ \mathbf{P}^n(K) & \text{if } x = y \end{cases}$$

where $< x, y >$ denotes the line spanned by the points x, y.

Let X be a closed subset of $\mathbf{P}^n(K)$. We set

$$\mathrm{Sec}_X^h = p_{12}^{-1}(X \times X \setminus \Delta_X),$$

$$Sec_X = \text{ closure of } \mathrm{Sec}_X^h \text{ in } Z.$$

The projection p_{12} and the projection $p_3 : Z \to \mathbf{P}^n(K)$ to the third factor define the regular maps

$$p : \mathrm{Sec}_X \to X \times X, \quad q : \mathrm{Sec}_X \to \mathbf{P}^n(K).$$

For any $(x, y) \in X \times X \setminus \Delta_X$ the image of the fibre $p^{-1}(x, y)$ under the map q is equal to the line $< x, y >$. Any such lines is called a *honest secant* of X. The union of all honest secants of X is equal to the image of Sec_X^h under the map q. The closure of this union is equal to $q(\mathrm{Sec}_X)$. It is denoted by $\mathrm{Sec}(X)$ and is called the *secant variety* of X.

Lemma 3. *Let X be an irreducible closed subset of $\mathbf{P}^n(K)$. The secant variety $Sec(X)$ is an irreducible projective algebraic set of dimension* $\leq 2\dim X + 1$.

Proof. It is enough to show that Sec_X^h is irreducible. This would imply that Sec_X and $\mathrm{Sec}(X)$ are irreducible, and by the theorem on dimension of fibres $\dim \mathrm{Sec}_X^h = \dim(X \times X) + 1 = 2\dim X + 1$. This gives

$$\dim \mathrm{Sec}(X) \leq \dim \mathrm{Sec}(X) = \dim \mathrm{Sec}_X^h = 2\dim X + 1.$$

To prove the irreduciblity of Sec_X^h we modify a little the proof of Lemma 2 of Lecture 12. We cannot apply it directly since Sec_X^h is not projective set. However, the map $p^h : \mathrm{Sec}_X^h \to X \times X \setminus \Delta_X$ is the restriction of the projection sets $(X \times X \setminus \Delta_X) \times \mathbf{P}^n(K) \to X \times X \setminus \Delta_X$. By Chevalley's Theorem from Lecture 9, the image of a closed subset of Sec_X^h is closed in $X \times X \setminus \Delta_X$. Only this additional property of the map $f : X \to Y$ was used in the proof of Lemma 2 of Lecture 12.

Let $X \subset \mathbf{P}^n(K)$ be a nonsingular projective set of dimension d and let $G(d+1, n+1)$ be the Grassmann variety of d-dimensional linear subspaces of $\mathbf{P}^n(K)$. Define the *Gauss map*

$$g : X \to G(d+1, n+1)$$

by sending a point x to the projective tangent space $ET(X)_x$. This is a regular map. In fact, it is easy to see that the Plücker coordinates of $ET(X)_x$ can be expressed via the polynomials $\frac{\partial F_i}{\partial T_j}(x)$ where $F_i = 0$ are defining equations of X.

Let

$$P = \{(x, E) \in \mathbf{P}^n(K) \times G(d+1, n+1) : x \in E\}.$$

The fibre of the second projection $pr_2 : P \to G(d+1, n+1)$ over the point $[E]$ representing a subspace E is isomorphic to E. Define the *tangential bundle* of X by

$$\mathrm{Tan}_X = \{(x, z) \in X \times P : g(x) = pr_2(z)\}.$$

It is a closed subset of $X \times P$ (equal to the pre-image of the diagonal $\Delta_{G(d+1,n+1)}$ under the map $g \times pr_2 : X \times P \to G(d+1, n+1) \times G(d+1, n+1)$). It is clear that under the the composition of the projections $\mathrm{Tan}(X) \to P \to \mathbf{P}^n(K)$ the fibre of the first projection $\mathrm{Tan}_X \to X$ over a point $x \in X$ is isomorphic to $ET(X)_x$. By Lemma 2, its dimension is equal to $\dim X$. From this we obtain that Tan_X is irreducible and of dimension $2\dim X$. Let $\mathrm{Tan}(X)$ be the image of Tan_X under the first projection map $P \to \mathbf{P}^n(K)$. It is equal to the union of embedded tangent spaces $ET(X)_x, x \in X$. We call it the *tangential variety* of X. We have proven

Lemma 4. *The tangential variety* $\mathrm{Tan}(X)$ *of an irreducible projective algebraic set of* $\mathbf{P}^n(K)$ *is an irreducible projective set of dimension* $\leq 2\dim X$.

Now everything is ready to prove the following main result of this Lecture:

Theorem 2. *Every nonsingular projective d-dimensional algebraic set* X *can be embedded into* $\mathbf{P}^{2d+1}$.

Proof. The idea is very simple. Let $X \subset \mathbf{P}^n(K)$, we shall try to project X into a lower-dimensional projective space. Assume $n > 2d+1$. Let $a \in \mathbf{P}^n(K) \setminus X$. By Lemma 1, the projection map

$$p_a : X \to Y \subset \mathbf{P}^{n-1}(K)$$

is an isomorphism unless either x lies on a honest secant of X or in the tangential variety of X. Since all honest secants are contained in the secant variety $\mathrm{Sec}(X)$ of X, and

$$\dim\mathrm{Sec}(X) \leq 2\dim X + 1 < n, \quad \dim\mathrm{Tan}(X) \leq 2\dim X < n,$$

we can always find a point $a \notin X$ for which the map p_a is an isomorphism. Continuing in this way, we prove the theorem.

Corollary. *Every projective algebraic curve (resp. surface) is isomorphic to a curve (resp. a surface) in* $\mathbf{P}^3(K)$ *(resp.* $\mathbf{P}^5(K)$*).*

Remark 2. The result stated in the Theorem is the best possible for projective sets. For example, the affine algebraic curve: $V(T_1^2 + F_n(T_2)) = 0$, where F_n is a polynomial of degree $n > 4$ without multiple roots, is not birationally isomorphic to any nonsingular plane projective algebraic curve. Unfortunately, we have no sufficient tools to prove this claim. Let me give one more unproven fact. To each nonsingular projective curve X one may attach an integer $g \geq 0$, called the *genus* of X. If $K = \mathbf{C}$ is the field of complex numbers, the genus is equal to the genus of the Riemann surface associated to X. Each compact Riemann surface is obtained in this way. Now for any plane curve $V(F) \subset \mathbf{P}^2(K)$ of degree n one computes the genus by the formula

$$g = \frac{(n-1)(n-2)}{2}.$$

Since some values of g cannot be realized by this formula (for example $g = 2, 4, 5$) we obtain that not every nonsingular projective algebraic curve is isomorphic to a plane curve.

Let $\mathrm{Sec}(X)$ be the secant variety of X. We know that it is equal to the closure of the union $\mathrm{Sec}(X)^h$ of honest secant lines of X. A natural guess is that the complementary set $\mathrm{Sec}(X) \setminus \mathrm{Sec}(X)^h$ consists of the union of tangent lines to X, or in other words to the tangential variety $\mathrm{Tan}(X)$ of X. This is true.

Theorem 3. *Let* $X \setminus \mathbf{P}^n(K)$ *be a nonsingular irreducible closed subset of* $\mathbf{P}^n(K)$. *Then*

$$\mathrm{Sec}(X) = \mathrm{Sec}(X)^h \cup \mathrm{Tan}(X).$$

Proof. Since $\mathrm{Sec}(X)$ is equal to the closure of $\mathrm{Sec}(X)^h$ it is enough to prove the following assertions:

(i) $\mathrm{Sec}(X)^h \cup \mathrm{Tan}(X)$ is a closed subset of $\mathbf{P}^n(K)$;

(ii) $\mathrm{Tan}(X) \subset \mathrm{Sec}(X)$.

Let $G = G(2, n+1)$ be the Grassmann variety of lines in $\mathbf{P}^n(K)$ and let

$$Q = \{(x, \ell) \in \mathbf{P}^n(K) \times G : x \in \ell\}.$$

Let $p : Q \to G$ be the second projection map $(x, \ell) \mapsto \ell$. Under the the first projection $q : Q \to \mathbf{P}^n(K)$, the fibre of p over a line $\ell \in G$ is mapped isomorphically to the line $\ell \subset \mathbf{P}^n(K)$. Let $J(X)$ be the set of lines which are either honest secants of X or tangent to X at some point of X. Suppose we prove that this set is a closed subset of G. Then its pre-image $p^{-1}(J(X))$ is closed and its image under q is closed (Chevalley's Theorem). But obviously this image is equal to the union $\mathrm{Sec}(X)^h \cup \mathrm{Tan}(X)$. This will prove assertion (i).

Let X be defined by homogeneous equations $F_1 = \ldots = F_N = 0$. There is no harm to assume that all of them of the same degree d. Let $\ell =< a, b >\in J(X)$ and $x = t_0 a + t_1 b \in \ell$ where t_0, t_1 projective coordiantes in ℓ. As we saw in the proof of Lemma 1 in Lecture 12 the coordinates of x are linear combination of the Plücker coordinates of ℓ. Thus if we restrict a homogeneous polynomial $F(T_0, \ldots, T_n)$ of degree m to ℓ we obtain a homogeneous polynomal in t_0, t_1 whose coefficients are homogenous polynomials of degree m in Plücker coordinates of ℓ. Now take any $m \geq d$ and consider the linear map

$$\Phi_\ell : K[T_0, \ldots, T_n]^N_{m-d} \to K[t_0, t_1]_m, \quad (G_1, \ldots, G_N \to \sum_{i=1}^{N} (G_i F_i)|\ell.$$

It is clear that $\ell \in J(X)$ if and only if the image of the map Φ_ℓ is contained in the subspace of homogeneous polynomials of degree m vanishing at the set $\ell \cap X$ of at least two points or having a multiple root at a point from $\ell \cap X$ (see Problem 6). Since the subset of $K[t_0, t_1]_m$ consisting of polynomials with at least two fixed roots counting with multiplicity is of codimension ≥ 2, we obtain that $\ell \in J(X)$ if and only if corank $\Phi_\ell \geq 2$. The latter condition is expressed by minors of the matrix of Φ_ℓ whose coefficients are polynomials in Plücker coordinates of ℓ. This proves that $J(X)$ is a closed subset of the Grassmannian and therefore proves assertion (i).

Now let us prove assertion (ii). We have to show that any polynomial $F(T_0, \ldots, T_n)$ vanishing on any honest secant of X must vanish on any line tangent to X. Intutively this is clear. If $< x, y >$ is a honest secant, we let y converge to x, the limit must be a tangent line ℓ at the point x. So, "by continuity", F must vanish at ℓ. Let us realize algebraically this idea.

Choose projective coordinates such that $x = (1, 0, \ldots, 0)$. Let $X_0 = X \cap (\mathbf{P}^n(K))_0$, and $(y_1, \ldots, y_n)$ be inhomogeneous coordinates of a point $y \in X_0$. The intersection of the line $< x, y >$ with X_0 consists of points $ty = (ty_1, \ldots, ty_n), t \in K$. Let

$$C = \{(y, t) \in \mathbf{A}^n(K)(K) \times \mathbf{A}^1(K) : ty \in X_0\}.$$

It consists of two irreducible components C_1 and C_2, where $C_1 = \{(y, 0) : y \in \mathbf{A}^n(K)\}$ and C_2 is the closure of the pre-image of $\mathbf{A}^1(K) \setminus \{0\}$ under the second projection $p_2 : C \to \mathbf{A}^1(K)$. It is clear that for any $(y, t) \in C_2$ with $t \neq 0$, we have $y \in \mathrm{Sec}(X)$. However if $t = 0$, the point y lies on a tangent line ℓ at x. In fact, the equation of C are $G_i(TZ_1, \ldots, TZ_n) = 0$, where $G_i(Z_1, \ldots, Z_n) = 0$ are the equations of X. We can write

$$G_i(Z_1, \ldots, Z_n) = \sum_{j=1}^{n} \frac{\partial G_i}{\partial Z_j}(0) Z_j + \text{higher terms in } Z_j,$$

hence

$$G_i(TZ_1,\ldots,TZ_n) = \sum_{j=1}^{n} T\frac{\partial G_i}{\partial Z_j}(0)Z_j + T \times \text{other stuff}.$$

This shows that the equation of C_1 is $T=0$, and the equation of the image of $C_2 \cup C_1$ in $\mathbf{A}^n(K)$ is equal to

$$\sum_{j=1}^{n} \frac{\partial G_i}{\partial Z_j}(0)Z_j = 0.$$

But this is exactly the equation of the embedded tangent space $ET(X)_x$. Now everything is clear. If a polynomial $F(Z_1,\ldots,Z_n)$ vanishes on the union of all secant lines spanned by x and ty for some $t \neq 0$, then $F(TZ_1,\ldots,TZ_n)$ vanishes on the whole component C_2, and hence $F(Z_1,\ldots,Z_n)$ vanishes on the union of tangent lines at x. Thus we have checked the second assertion and the theorem is proven.

Remark 3. If X is singular, the right analog of the embedded tangent space $ET(X)$ is the tangent cone $CT(X)_x$. It is defined as the the union of limits of the lines $< x, y >$ where $y \in X$. The proof of Theorem 3 shows that $CT(X)_x$ is given by homogeneous equations which are defined by initial forms of affine equations defining X in an affine open neighborhood of the point x. See details in Shafarevich's book, Chapter II, §1, section 5.

Definition A closed subset $X \subset \mathbf{P}^n(K)$ is called *non-degenerate* if it is not contained in a hyperplane in $\mathbf{P}^n(K)$. A nondegenerate subset is called *linearly normal* if it cannot be obtained as an isomorphic projection of some $X' \subset \mathbf{P}^{n+1}(K)$.

Theorem 4. *Let X be a nonsingular irreducible non-degenerate projective curve in $\mathbf{P}^3(K)$. Then X cannot be isomorphically projected into $\mathbf{P}^2(K)$ from a point outside X. In particular any plane nonsingular projective curve of degre > 1 is linearly normal.*

Proof. Applying Theorem 3 and Lemma 1, we have to show that $\mathrm{Sec}(X) = \mathbf{P}^3(K)$. Assume the contrary. Then $\mathrm{Sec}(X)$ is an irreducible surface. For any $x \in X, \mathrm{Sec}(X)$ contains the union of lines joining x with some point $y \neq x$ in X. Since X is not a line, the union of lines $< x, y >, y \in Y, y \neq x$, is of dimension > 1 hence equal to $\mathrm{Sec}(X)$. Pick up three non-collinear points $x, y, z \in X$. Then $\mathrm{Sec}(X)$ contains the line $< x, y >$. Since each point of $\mathrm{Sec}(X)$ is on the line passing through z, we obtain that each line $< z, t >, t \in < x, y >$ belongs to $\mathrm{Sec}(X)$. But the union of these lines is the plane spanned by x, y, z. Thus $\mathrm{Sec}(X)$ coincides with this plane. Since X is obviously contained in $\mathrm{Sec}(X)$ this is absurd.

The next two important results of F. Zak are given without proof.

Theorem 5. *Let X be a nonsingular nondegenerate closed irreducible subset of $\mathbf{P}^n(K)$ of dimension d. Assume $\mathrm{Sec}(X) \neq \mathbf{P}^n(K)$. Then*

$$n \geq 2 + \frac{3d}{2}.$$

In particular, any nonsingular nondegenerate d-dimensional closed subset of $\mathbf{P}^n(K)$ is linearly normal if $n \leq \frac{3d}{2}$.

If $d = 2$, this gives that any surface of degre > 1 in $\mathbf{P}^3(K)$ is linearly normal. This bound is sharp. To show this let us consider the Veronese surface $X = v_2(\mathbf{P}^2(K)$ in $\mathbf{P}^5(K)$. Then we know that it is isomorphic to the set of symmetric 3×3-matrices of rank 1 up to proportionality. It is easy to see, by using linear algebra, that $\mathrm{Sec}(X)$ is equal to the set of symmetric matrices of rank ≤ 2 up to proportionality. This is a cubic hypersurface in $\mathbf{P}^3(K)$ defined by the equation expresing the determinant of symmetric matrix. Thus we can isomorphically project X in $\mathbf{P}^4(K)$.

Remark 4. According to a conjecture of R. Hartshorne, any non-degenerate nonsingular closed subset $X \subset \mathbf{P}^n(K)$ of dimension $d > 2n/3$ is a complete intersection (i.e. can be given by $n-d$ homogeneous equations).

Definition. A *Severi variety* is a nonsingular irreducible algebraic set X in $\mathbf{P}^n(K)$ of dimension $d = 2(n-2)/3$ which is not contained in a hyperplane and with $\mathrm{Sec}(X) \neq \mathbf{P}^n(K)$.

The following result of F. Zak classifies Severi varieties in characteristic 0:

Theorem 6. *Assume char$(K) = 0$. Each Severi variety is isomorphic to one of the following four varieties:*
($n = 2$) the Veronese surface $v_2(\mathbf{P}^2(K)) \subset \mathbf{P}^5(K)$;
($n = 4$) the Segre variety $s_{2,2}(\mathbf{P}^2(K) \times \mathbf{P}^2(K)) \subset \mathbf{P}^8(K)$;
($n = 8$) the Grassmann variety $G(2,6) \subset \mathbf{P}^{14}(K)$ of lines in $\mathbf{P}^5(K)$;
($n = 16$) the E_6-variety X in $\mathbf{P}^2(K)$.

The last variety is defined as follows. Choose a bijection between the set of 27 lines on a nonsingular cubic surface and variables $T_0, \ldots, T_{26}$. For each triple of lines which span a tritangent plane form the corresponding monomial $T_iT_jT_k$. Let F be the sum of such 45 monomials. Its set of zeroes in $\mathbf{P}^{26}(K)$ is a cubic hypersurface $Y = V(F)$. Then X is equal to the set of singularities of Y (it is the set of zeroes of 27 partial derivatives of F) and Y equals $\mathrm{Sec}(X)$. From the point of view of the group theory, $X = G/P$, where G is a simply connected simple algebraic linear group of exceptional type E_6, and P its maximal parabolic subgroup corresponding to the dominant weight ω defined by the extreme vertex of one of the long arms of the Dynkin diagram of the root system of G. The space $\mathbf{P}^{26}(K)$ is the projectivization of the representation of G with highest weight ω.

There is a beautiful uniform description of the four Severi varieties. Recall that a composition algebra is a finite-dimensional algebra A over a field K (not necessary commutative or associative) such that there exists a non-degenerate quadratic form $\Phi : A \to K$ such that for any $x, y \in A$

$$\Phi(x \cdot y) = \Phi(x)\Phi(y).$$

According to a classical theorem of A. Hurwitz there are four isomorphism classes of composition algebras over a field K of characteristic 0: K, Co, Ha and Oc of dimension $1, 2, 4$ and 8, respectively. Here

$$Co = K \oplus K, (a,b) \cdot (a',b') = (aa' - bb', ab' + a'b),$$

$$Ha = Co \oplus Co, (x,y) \cdot (x',y') = (x \cdot x' - \bar{y} \cdot y', x \cdot y' + y \cdot \bar{x}'),$$

$$Oc = Ha \oplus Ha, (h,g) \cdot (h',g') = (h \cdot h' - \bar{g} \cdot g', h \cdot g' + g \cdot \bar{h}'),$$

where for any $x = (a,b) \in Co$ we set $\bar{x} = (a, -b)$, and for any $h = (x,y) \in Ha$ we set $\bar{h} = (\bar{x}, -y)$. The quadratic form Φ is given by

$$\Phi(x) = x \cdot \bar{x},$$

where $\bar{x}$ is defined as above for $A = Ca$ and H, $\bar{x} = x$ for $A = K$, and $\bar{x} = (\bar{h}, -h')$ for any $x = (h,h') \in Oc$.

For example, if $K = \mathbf{R}$, then $Co \cong \mathbf{C}$ (complex numbers), $Ha \cong \mathbf{H}$ (quaternions), $Oc = \mathbf{O}$ (octonians or Cayley numbers).

For every composition algebra A we can consider the set $\mathcal{H}_3(A)$ of Hermitian 3×3-matrices (a_{ij}) with coefficients in A, where Hermitian means $a_{ij} = \bar{a}_{ji}$. Its dimension as a vector space over K equals $3 + 3r$, where $r = \dim_K A$. There is a natural definition of the rank of a matrix from $\mathcal{H}_3(A)$. Now Theorem 6 says that the four Severi varieties are closed subsets of $\mathbf{P}^{3r+2}$ defined by rank 1 matrices in $\mathcal{H}_3(A)$. The corresponding secant variety is defined by the homogeneous cubic form representing the "determinant" of the matrix.

Let us define $\mathbf{P}^n(A)$ for any composition algebra as $A^{n+1} \setminus \{0\}/A^*$. Then one view the four Severi varieties as the "Veronese surfaces" corresponding to the projective planes over the four composition algebra.

As though it is not enough of these mysterious coincidences of the classifications, we add one more. Using the stereographic projection one can show that

$$\mathbf{P}^1(\mathbf{R}) = S^1, \mathbf{P}^1(\mathbf{C}) = S^2, \mathbf{P}^1(\mathbf{H}) = S^4, \mathbf{P}^1(\mathbf{O}) = S^8,$$

where S^k denote the unit sphere of dimension k. The canonical projection

$$A^2 \setminus \{0\} \to \mathbf{P}^1(A) = S^k$$

restricted to the subset $\{(x, y) \in R^2 : x \cdot \bar{x} + y \cdot \bar{y} = 1\} = S^{2r-1}$ defines a map

$$\pi : S^{2r-1} \to S^r$$

which has a structure of a smooth bundle with fibres diffeomorphic to the sphere $S^{r-1} = \{x \in A^* : x \cdot \bar{x} = 1\}$. In this way we obtain 4 examples of a Hopf bundle: a smooth map of a sphere to a sphere which is a fibre bundle with fibres diffeomorphic to a sphere. According to a famous result of F. Adams, each Hopf bundle is diffeomorphic to one of the four examples coming from the composition algebras.

Is there any direct relationship between Hopf bundles and Severi varieties?

Problems.

1. Let $X = v_3(\mathbf{P}^1(K))$ be a twisted cubic in $\mathbf{P}^3(K)$. Show that the variety $J(X) \subset G(2,4)$ of honest secants and tangents of X is isomorphic to a the projective plane.
2. Find the equation of the tangential surface $\mathrm{Tan}(X)$ of the twisted cubic curve in $\mathbf{P}^3(K)$.
3. Show that the secant variety $\mathrm{Sec}(X)$ of the Veronese curve $X = v_4(\mathbf{P}^1(K))$ in $\mathbf{P}^4(K)$ is a cubic hypersurface which is singular along X. Find the equations of the tangential variety $\mathrm{Tan}(X)$.
4. Verify that the Segre variety $s_{2,2}(\mathbf{P}^2(K) \times (\mathbf{P}^2(K)$ is a Severi variety.
5. Assume that the secant variety $\mathrm{Sec}(X)$ is not the whole space. Show that any X is contained in the set of singular points of $\mathrm{Sec}(X)$.
6. Show that a line ℓ is tangent to an algebraic set X at a point $x \in X$ if and only if the restriction to ℓ of any polynomial vanishing on X has the point x as its multiple root.

7*. Let X be a nonsingular irreducible projective curve in $\mathbf{P}^n(K)$. Show that the image of the Gauss map $g : X \to G(2, n+1)$ is birationally isomorphic to X unless X is a line.

Lecture 16. BLOWING UP AND RESOLUTION OF SINGULARITIES

Let us consider the projection map $p_a : \mathbf{P}^n(K) \setminus \{a\} \to \mathbf{P}^{n-1}(K)$. If $n > 1$ it is impossible to extend it to the point a. However, we may try to find another projective set X which contains an open subset isomorphic to $\mathbf{P}^n(K) \setminus \{a\}$ such that the map p_a extends to a regular map $\bar{p}_a : X \to \mathbf{P}^{n-1}(K)$. The easiest way to do it is to consider the graph $\Gamma \subset \mathbf{P}^n(K) \setminus \{a\} \times \mathbf{P}^{n-1}(K)$ of the map p_a and take for X its closure in $\mathbf{P}^n(K) \times \mathbf{P}^{n-1}(K)$. The second projection map $X \to \mathbf{P}^{n-1}(K)$ will solve our problem. It is easy to find the bi-homogeneous equations defining X. For simplicity we may assume that $a = (1, 0, \ldots, 0)$ so that the map p_a is given by the formula $(x_0, x_, \ldots, x_n) \to (x_1, \ldots, x_n)$. Let $Z_0, \ldots, Z_n$ be projective coordinates in ψ and let $T_1, \ldots, T_n$ be projective coordinates in $\mathbf{P}^{n-1}(K)$. Obviously the graph Γ is contained in the closed set X defined by the equations

$$(*) \qquad Z_iT_j - Z_jT_i = 0, i, j = 1, \ldots, n.$$

The projection $q : X \to \mathbf{P}^{n-1}(K)$ has the fibre over a point $t = (t_1, \ldots, t_n)$ equal to the linear subspace of $\mathbf{P}^n(K)$ defined by the equations

$$(**) \qquad Z_it_j - Z_jt_i = 0, i, j = 1, \ldots, n.$$

Assume that $t_i = 1$. Then the matrix of coefficients of the system of linear equations $(**)$ contains $n-1$ unit columns so that its rank is equal to $n-1$. This shows that the fibre $q^{-1}(t)$ is isomorphic, under the first projection $X \to \mathbf{P}^n(K)$, to the line spanned by the points $(0, t_1, \ldots, t_n)$ and $(1, 0, \ldots, 0)$. On the other hand the first projection is an isomorphism over $\mathbf{P}^n(K) \setminus \{0\}$. Since X is irreducible (all fibres of q are of the same dimension), we obtain that X is equal to the closure of Γ. By plugging $z_1 = \ldots z_n$ in equations $(**)$ we see that the fibre of p over the point $a = (1, 0, \ldots, 0)$ is isomorphic to the projective space $\mathbf{P}^{n-1}(K)$. Under the map q this fibre is mapped isomorphically to $\mathbf{P}^{n-1}(K)$.

The pre-image of the subset $\mathbf{P}^n(K) \setminus V(Z_0) \cong \mathbf{A}^n(K)$ under the map p is isomorphic to the closed subvariety B of $\mathbf{A}^n(K) \times \mathbf{P}^{n-1}(K)$ given by the equations $(*)$ where we consider $Z_1, \ldots, Z_n$ as inhomogeneous coordinates in affine space. The restriction of the map p to B is a regular map $\sigma : B \to \mathbf{A}^n(K)$ satisfying the following properties

(i) $\sigma|\sigma^{-1}(\mathbf{A}^n(K) \setminus \{(0, \ldots, 0)\}) \to \mathbf{A}^n(K) \setminus \{(0, \ldots, 0)\}$ is an isomorphism;

(ii) $\sigma^{-1}(0, \ldots, 0) \cong \mathbf{P}^{n-1}(K)$.

We express this by saying that σ " blows up" the origin. Of course if we take $n = 1$ nothing happens. The algebraic set B is isomorphic to $\mathbf{A}^n(K)$. But if take $n = 2$, then B is equal to the closed subset of $\mathbf{A}^2(K) \times \mathbf{P}^1(K)$ defined by the equation

$$Z_2T_0 - T_1Z_1 = 0.$$

It is equal to the union of two affine algebraic sets V_0 and V_1 defined by the condition $T_0 \neq 0$ and $T_1 \neq 0$, respectively. We have

$$V_0 = V(Z_2 - XZ_1) \subset \mathbf{A}^2(K) \times \mathbf{P}^1(K)_0, \quad X = T_1/T_0,$$

$$V_1 = V(Z_2Y - Z_1) \subset \mathbf{A}^2(K) \times \mathbf{P}^1(K)_1, \quad Y = T_0/T_1.$$

If $L : Z_2 - tZ_1 = 0$ is the line in $\mathbf{A}^2(K)$ through the origin "with slope" t, then the pre-image of this line under the projection $\sigma : B \to \mathbf{A}^2(K)$ consists of the union of two curves, the fibre $E \cong \mathbf{P}^1(K)$ over the origin, and the curve $\bar{L}$ isomorphic to L under σ. The curve $\bar{L}$ intersects E at the point $((0,0),(1,t)) \in V_0$. The pre-image of each line L with the equation $tZ_2 - Z_1$ consists of E and the curve intersecting E at the point $((0,0),(t,1)) \in V_1$. Thus the points of E can be thought as the set of slopes of the lines through $(0,0)$. The "infinite slope" corresponding to the line $Z_1 = 0$ is the point $(0,1) \in V_1 \cap E$.

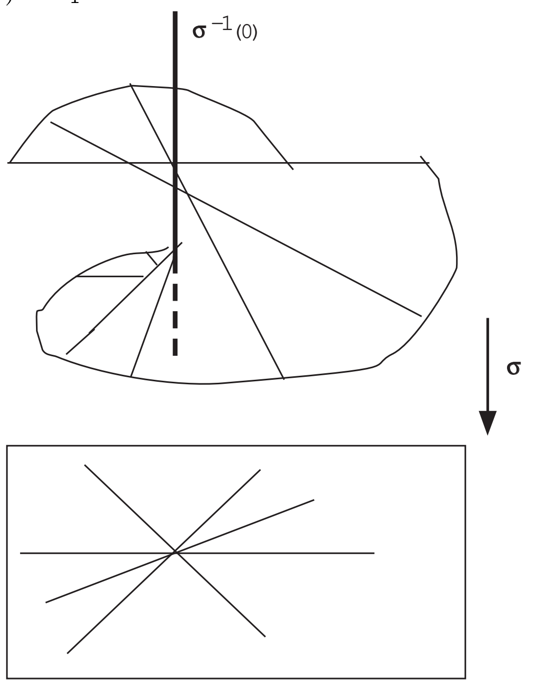

Fig.1

Let I be an ideal in a commutative ring A. Each power I^n of I is a A-module and $I^nI^r \subset I^{n+r}$ for every $n, r \geq 0$. This shows that the multipication maps $I^n \times I^r \to I^{n+r}$ define a ring structure on the direct sum of A-modules

$$A(I) = \oplus_{n \geq 0} I^n.$$

Moreover, it makes this ring a graded algebra over $A = A(I)_0 = I^0$. Its homogeneous elements of degree n are elements of I^n.

Assume now that I is generated by a finite set $f_0, \ldots, f_n$ of elements of A. Consider the surjective homomorphism of graded A-algebras

$$\phi : A[T_0, \ldots, T_n] \to A(I)$$

defined by sending T_i to f_i. The kernel $Ker(\phi)$ is a homogeneous ideal in $A[T_0, \ldots, T_n]$. If we additionally assume that A is a finitely generated algebra over a field k, we can interpret $Ker(\phi)$

as the ideal defining a closed subset in the product $X \times \mathbf{P}_k^n$ where X is an affine algebraic variety with $\mathcal{O}(X) \cong A$. Let Y be the subvariety of X defined by the ideal I.

Definition The subvariety of $X \times \mathbf{P}_k^n$ defined by the ideal $\mathrm{Ker}(\phi)$ is denoted by $B_Y(X)$ and is called the blow-up of X along Y. The morphism $\sigma : B_Y(X) \to X$ defined by the projection $X \times \mathbf{P}_k^n \to X$ is called the *monoidal transformation* or the *σ-process* or the *blowing up morphism* along Y.

Let us fix an algebraically closed field K containing k and describe the algebraic set $B_Y(X)(K)$ as a subset of $X(K) \times \mathbf{P}^n(K)$. Let $U_i = X \times (\mathbf{P}^n(K))_i$ and $B_Y(X)_i = B_Y(X) \cap U_i$. This is an affine algebraic k-set with

$$\mathcal{O}(B_Y(X)_i) \cong \mathcal{O}(X)[T_0/T_i \ldots, T_n/T_i]/Ker(\phi)_i$$

where $Ker(\phi)_i$ is obtained from the ideal $Ker(\phi)$ by dehomogenization with respect to the variable T_i. The fact that the isomorphism class of $B_Y(X)$ is independent of the choice of generators $f_0, \ldots, f_n$ follows from the following

Lemma 1. *Let $Y \subset X \times \mathbf{P}_k^n(K)$ and $Y' \subset X \times \mathbf{P}_k^r(K)$ be two closed subsets defined by homogeneous ideals $I \subset \mathcal{O}(X)[T_0, \ldots, T_n]$ and $J \subset \mathcal{O}(X)[T_0', \ldots, T_r']$, respectively. Let $p : Y \to X$ and $p' : Y' \to X$ be the regular maps induced by the first projections $X \times \mathbf{P}_k^n(K) \to X$ and $X \times \mathbf{P}_k^r(K) \to X$. Assume that there is an isomorphism of graded $\mathcal{O}(X)$-algebras $\psi : \mathcal{O}(X)[T_0', \ldots, T_r']/I' \to \mathcal{O}(X)[T_0, \ldots, T_n]/I$. Then there exists an isomorphism $f : Y \to Y'$ such that $p = p' \circ f$.*

Proof. Let $t_i' = T_i' \bmod I', t_i = T_i \bmod I$, and let

$$\psi(t_i') = F_i(t_1, \ldots, t_n), i = 0, \ldots, r,$$

for some polynomial $F_i[T_0, \ldots, T_n]$. Since f is an isomorphism of graded $\mathcal{O}(X)$-algebras the polynomials $F_i(T)$ are linear and its coefficients are regular functions on X. The value of F_i at a point $(x, t) = (x, (t_0, \ldots, t_n))$ in $X \times \mathbf{P}_k^n(K)$ is defined by plugging x into the coefficients and plugging t into the unknowns T_j. Define $f : X \to Y$ by the formula:

$$f(x, t) = (x, (F_0(x, t), \ldots, F_n(x, t))).$$

Since ψ is invertible, there exist linear polynomials $G_j(T) \in \mathcal{O}(X)[T_0', \ldots, T_r'], j = 0, \ldots, n$, such that

$$F_i(G_0(t_0', \ldots, t_n'), \ldots, G_n(t_0, \ldots, t_n')) = t_i', i = 0, \ldots, r,$$

$$G_j(F_0(t_0, \ldots, t_n), \ldots, F_n(t_0, \ldots, t_n)) = t_j, j = 0, \ldots, n.$$

This easily implies that f is defined everywhere and is invertible. The property $p = p' \circ f$ follows from the definition of f.

Example 1. We take $X = \mathbf{A}_k^2(K), \mathcal{O}(X) = k[Z_1, Z_2], I = (Z_1, Z_2), Y = V(I) = \{(0, 0)\}$. Then $\phi : k[Z_1, Z_2][T_0, T_1] \to k[Z_1, Z_2](I)$ is defined by sending T_0 to Z_1, and T_1 to Z_2. Obviously $\mathrm{Ker}(\phi)$ contains $Z_2T_0 - Z_1T_1$. We will prove later in Proposition 2 that $\mathrm{Ker}(\phi) = (Z_2T_0 - Z_1T_1)$. Thus $B_Y(X)$ coincides with the example considered in the beginning of the Lecture.

Lemma 2. *Let $U = D(f) \subset X$ be a principal affine open subset of an affine set X, then*

$$B_{Y\cap U} \cong \sigma^{-1}(U).$$

Proof. We have $\mathcal{O}(U) \cong \mathcal{O}(X)_f, I(Y \cap U) = I(Y)_f$. If $I(Y)$ is generated by $f_0, \ldots, f_n$ then $I(Y \cap U)$ is generated by $f_0/1, \ldots, f_n/1$, hence $B_{Y\cap U}$ is defined by the kernel of the homomorphism

$$\phi_f : \mathcal{O}(X)_f[T_0, \ldots, T_n] \to \mathcal{O}(X)(I(Y)_f), T_i \to f_i/1.$$

Obviously the latter is obtained by localizing the homomorphism of $\mathcal{O}(X)$-algebras

$$\phi : \mathcal{O}(X)[T_0, \ldots, T_n] \to \mathcal{O}(X)(I(Y)), T_i \to f_i.$$

Therefore the kernel of ϕ_f is isomorphic to $(\mathrm{Ker}(\phi))_f$. The set of zeroes of this ideal is equal to $\sigma^{-1}(D(f))$.

Proposition 1. *The blow-up $\sigma : B_Y(X) \to X$ induces an isomorphism*

$$\sigma^{-1}(X \setminus Y) \cong X \setminus Y.$$

Proof. It is enough to show that for any prinicpal open subset that $U = D(f) \subset X \setminus Y$ the induced map $\sigma^{-1}(U) \to U$ is an isomorphism. Since $Y \subset X \setminus U$ and $I(Y)$ is radical ideal, f must belong to $I(Y)$. Thus $I(Y)_f = \mathcal{O}(X)_f$ and, taking 1 as a generator of $I(Y)_f$ we get $\mathcal{O}(X)_f((1) = \mathcal{O}(X)_f$, and the map $\phi_f : \mathcal{O}(X)_f[T_0] \to \mathcal{O}(X)_f, T_0 \to 1$ has the kernel equal to $(T_0 - 1)$. Applying the previous Lemma, we get $B_\emptyset(X) \cong D(f) \cong \sigma^{-1}(D(f))$. This proves the assertion.

To find explicitly the equations of the blow-up $B_Y(X)$, we need to make some assumptions on X and Y.

Definition. Let A be a commutative ring. A sequence of elements $a_1, \ldots, a_n \in A$ is called a *regular sequence* if the ideal generated by $a_1, \ldots, a_n$ is a proper ideal of A and, for any $i = 1, \ldots, n$, the image of a_i in $A/(a_1, \ldots, a_{i-1})$ is a non-zero divisor (we set $a_0 = 0$).

Lemma 3. *Let M be a module overa commutative ring A. Assume that for any maximal ideal m of A, the localization $M_m = \{0\}$. Then $M = \{0\}$.*

Proof. Let $x \in M$. For any maximal ideal $m \subset A$, there exists $a_m \notin m$ such that $a_m x = 0$. The ideal of A generated by the elements a_m is the unit ideal. Hence $1 = \sum_m b_m a_m$ for some $b_m \in A$ and

$$x = 1 \cdot x = \sum_m b_m a_m x = 0.$$

This proves the assertion.

Proposition 2. *Let $a_0, \ldots, a_n$ be a regular sequence of elements in an integral domain A and let I be the ideal generated by $a_1, \ldots, a_n$. Then the kernel J of the homomorphism*

$$\phi : A[T_0, \ldots, T_n] \to A(I), T_i \mapsto a_i,$$

is generated by the polynomials $P_{ij} = a_i T_j - a_j T_i, i, j = 0, \ldots, n$.

Proof. Let J' be the ideal in $A[T_0, \ldots, T_n]$ generated by the polynomials P_{ij}. Let $A_0 = A[a_0^{-1}] \cong A_{a_0}$ be the subring of the quotient field $Q(A)$ of A, $I_0 = (a_1/a_0, \ldots, a_n/a_0) \subset A_0$.

Define a homomorphism $\phi_0 : A[Z_1, \ldots, Z_n] \to A_0[I_0]$ via sending each Z_i to a_i/a_0. We claim that $J_0 = \mathrm{Ker}(\phi_0)$ is equal to the ideal J_0' generated by the polynomials $L_i = a_0Z_i - a_i$. Assume this is so. Then for any $F(T_0, \ldots, T_n) \in Ker(\phi)$, after dehomogenizing with respect to T_0, we obtain that $F(1, Z_1, \ldots, Z_n)$ belongs to J_0'. This would immediately imply that $T_0^N F \in J'$ for some $N \geq 0$. Replacing T_0 with T_i, and f_0 with f_i, we will similarly prove that $T_i^N F \in J'$ for any $i = 0, \ldots, n$. Now consider the A-submodule M of $A[T_0, \ldots, T_n]/J'$ generated by F. Since $T_i^N F = 0, i = 0, \ldots, n$, it is a finitely generated A-module. For any maximal ideal $m \subset A$ let $\bar{P}_{ij} = (a_i \bmod m)T_j - (a_j \bmod m)T_i$. The ideal in $(A/m)[T_0, \ldots, T_n]$ generated by the linear polynomials $\bar{P}_{ij}$ is obviously prime. Thus $T_i^N F = 0$ implies $M \otimes A/m = \{0\}$. Applying Nakayama's Lemma we infer that, for any maximal ideal $m \subset A$, the localization M_m is equal to zero. By the previous lemma this gives $M = 0$ so that $F \in J'$.

It remains to show that $\mathrm{Ker}(\phi_0)$ is generated by by the polynomials $L_i = a_0Z_i - a_i$. We use induction on n. Assume $n = 1$. Let $F \in \mathrm{Ker}(\phi_0)$, i.e., $\phi_0(F(Z_1)) = F(a_1/a_0) = 0$. Dividing by $L_1 = a_0Z_1 - a_1$, we obtain for some $G(Z_1) \in A[Z_1]$ and $r \geq 0$

$$a_0^r F(Z_1) = G(Z_1)(a_0Z_1 - a_1) = a_0G(Z_1)Z_1 - a_1G(Z_1).$$

Since (a_0, a_1) is a regular sequence, this implies that $G(a) \in (a_0)$ for any $a \in A$. From this we deduce that all coefficients of $G(Z_1)$ are divisible by a_0 so that we can cancel a_0 in the previous equation. Proceeding in this way we find, by induction on r, that F is divisible by L_1.

Now assume $n > 1$ and consider the map ϕ_0 as the composition map

$$A[Z_1, \ldots, Z_n] \to A'[Z_2, \ldots, Z_n] \to A_0[I_0] = A'[I'],$$

where $A' = A[a_1/a_0]$ is the subalgebra of A_0 generated by a_1/a_0, and $I' = (a_2/a_0, \ldots, a_n/a_0)$. It is easy to see that $a_0, \ldots, a_n$ is a regular sequence in A'. By induction, $L_2, \ldots, L_n$ generate the kernel of the second map $A'[Z_2, \ldots, Z_n] \to A_0[I_0]$. Thus $F(Z_1, \ldots, Z_n) \in Ker(\phi_0)$ implies

$$F(a_1/a_0, Z_2, \ldots, Z_n) = \sum_{i=2}^{n} Q_i(a_1/a_0, Z_2, \ldots, Z_n)L_i,$$

for some polynomials $Q_i(Z_1, \ldots, Z_n) \in A[Z_1, \ldots, Z_n]$. Thus by the case $n = 1$

$$F(Z_1, \ldots, Z_n) - \sum_{i=2}^{n} Q_i(a_1/a_0, Z_2, \ldots, Z_n)L_i \in (L_1),$$

and we are done.

Example 2. Take $A = k[Z_1, \ldots, Z_N], I = (a_0, \ldots, a_n) = (Z_1, \ldots, Z_{n+1})$ to obtain that the blow-up $B_{V(I)}(\mathbf{A}_k^N)(K))$ is a subvariety of $\mathbf{A}_k^N \times \mathbf{P}_k^n$ given by the equations

$$T_0Z_i - T_{i-1}Z_1 = 0, i = 1, \ldots, n+1.$$

This agrees with Example 1.

Remark 1. The assertion of Proposition 2 can be generalized as follows. Let $a_1, \ldots, a_n$ be a regular sequence in A. Consider the free module A^n with basis $e_1, \ldots, e_n$ and let $\bigwedge^r A^n$ be its r-th exterior power. It is a free A-module with basis formed by the wedge products $e_{i_1} \wedge \ldots \wedge e_{i_r}$ where $1 \leq i_1 < \ldots, i_r \leq n$. For each $r = 1, \ldots, n$. Define the map

$$\delta_r : \bigwedge^r A^n \to \bigwedge^{r-1} A^n$$

by the formula

$$\delta_r(e_{i_1} \wedge \ldots \wedge e_{i_r}) = \sum_i (-1)^j a_{i_j} e_{i_1} \wedge \ldots \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \ldots \wedge e_{i_r}.$$

Now the claim is that the complex of A-modules (called the *Koszul complex*)

$$\{0\} \to \bigwedge^n A^n \to \bigwedge^{n-1} A^n \to \ldots \to \bigwedge^2 A^n \to \bigwedge^1 A^n \to A \to A/(a_1, \ldots, a_n) \to \{0\}$$

is exact. The previous proposition asserts only that this complex is exact at the term $\bigwedge^1 A^n$.

Proposition 3. *Let X be an affine irreducible algebraic k-set, I be an ideal in $\mathcal{O}(X)$ generated by a regular sequence $(f_0, \ldots, f_n)$, and let $Y = V(I)$ be the set of zeroes of this ideal. Let $\sigma : B_Y(X) \to X$ be the blow-up of X along Y. Then for any $x \in Y$,*

$$\sigma^{-1}(x) \cong \mathbf{P}^n(K).$$

The pre-image of every irreducible component of Y is an irreducible subset of $B_Y(X)$ of codimension 1.

Proof. By Proposition 2, $Z = B_Y(X)$ is a closed subset of $X \times \mathbf{P}^n(K)$ defined by the equations

$$T_0 f_i - T_i f_0 = 0, i = 1, \ldots, n.$$

For any point $y \in Y$ we have $f_0(y) = \ldots = f_n(y) = 0$. Hence for any $t \in \mathbf{P}^n(K)$, the point (y, t) is a zero of the above equations. This shows that $\sigma^{-1}(y)$ is equal to the fibre of the projection $X \times \mathbf{P}^n(K) \to X$ over y which is obviously equal to $\mathbf{P}^n(K)$. For each irreducible component Y_i of Y the restriction map $\sigma : \sigma^{-1}(Y_i) \to Y_i$ has fibres isomorphic to n-dimensional projective spaces. By Lemma 2 of Lecture 12 (plus the remark made in the proof of Lemma 3 in Lecture 15) we find that $\sigma^{-1}(Y_i)$ is irreducible of dimension equal to $n + \dim Y_i$. By Krull's Hauptidealsatz, $\dim Y_i = \dim X - n - 1$ (here we use again that $(f_0, \ldots, f_n)$ is a regular sequence).

Lemma 4. *Let X be a nonsingular irreducible affine algebraic k-set, Y be a nonsingular closed subset of X. For any $x \in Y$ with $\dim_x Y = \dim_x X - n$ there exists an affine open neighborhood U of x in X such that $Y \cap U = V(f_1, \ldots, f_n)$ for some regular sequence $(f_1, \ldots, f_n)$ of elements in $\mathcal{O}(U)$.*

Proof. Induction on n. The case $n = 1$ has been proven in Lecture 13. Let $f_0 \in I(Y)$ such that its germ $(f_0)_x$ in $m_{X,x}$ does not belong to $m_{X,x}^2$. Let $Y' = V(f_0)$. By Lemma 2 from lecture 14, $T(Y')_x$ is of codimension 1 in $T(X)_x$. By Krull's Hauptidealsatz, $\dim_x Y' = \dim X - 1$, hence Y' is nonsingular at x. Replacing X with a smaller open affine set U, we may assume that $Y' \cap U$ is nonsingular everywhere. By induction, for some $V \subset Y', Y \cap V$ is given in V by an ideal $(f_1, \ldots, f_n)$ so that Y is given locally by the ideal $(f_0, \ldots, f_n)$. Now the assertion follows from the following statement from Commutative Algebra (see Matsumura, pg.105): A sequence $(a_1, \ldots, a_n)$ of elements from the maximal ideal of a regular local ring A is a regular sequence if and only if $\dim A/(a_1, \ldots, a_n) = \dim A - n$. By this result, the germs of $f_0, \ldots, f_n$ in $\mathcal{O}_{X,x}$ form a regular sequence. Then it is easy to see that their representatives in some $\mathcal{O}(U)$ form a regular sequence.

Theorem 1. *Let $\sigma : B_Y(X) \to X$ be the blow-up of a nonsingular irreducible affine algebraic k-set X along a nonsingular closed subset Y. Then the following is true*
(i) σ is an isomorphism outside Y;
(ii) $B_Y(X)$ is nonsingular;
(iii) for any $y \in Y, \sigma^{-1}(y) \cong \mathbf{P}^n(K)$, where $n = \mathrm{codim}_y(Y,X) - 1 = \dim X - \dim_y Y - 1$;
(iv) for any irreducible component Y_i of Y, $\sigma^{-1}(Y_i)$ is an irreducible subset of codimension one.

Proof. Properties (i) and (iv) have been already verified. Propertry (iii) follows from Proposition 3 and lemma 4. We include them only for completeness sake. Using (i), we have to verify the nonsingularity of $B_Y(X)$ only at points x' with $\sigma(x') = y \in Y$. Replacing X by an open affine neighborhood U of y, we may assume that $Y = V(I)$ where I is an ideal generated by a regular sequence $f_0, \ldots, f_n$. By Lemma 2, $\sigma^{-1}(U) \cong B_{Y \cap U}(U)$ so that we may assume $X = U$. By Proposition 2, $B_Y(X) \subset X \times \mathbf{P}^n_k(K)$ is given by the equations: $f_iT_j - f_jT_i = 0, i,j = 0, \ldots, n$. Let $p = (y,t) \in B_Y(X)$ where $y \in Y, t = (t_0, \ldots, t_n) \in \mathbf{P^n(K)}$. We want to verify that it is a nonsingular point of $B_Y(X)$. Without loss of generality we may assume that the point p lies in the open subset $W = B_Y(X)_0$ where $t_0 \neq 0$. Since

$$T_0(f_iT_j - f_jT_i) = T_i(f_0T_j - f_jT_0) - T_j(f_0T_i - f_iT_0)$$

we may assume that $B_Y(X)$ is given by the equations

$$f_0T_i - f_iT_0 = 0, i = 0, \ldots, n$$

in an affine neighborhood of the point p. Let $G_1(T_1, \ldots, T_N) = \ldots = G_m(T_1, \ldots, T_N)$ be the system of equations defining X in $\mathbf{A}^N(K)$ and let $F_i(T_1, \ldots, T_N)$ represent the function f_i. Then W is given by the following equations in $\mathbf{A}^N(K) \times \mathbf{A}^n(K)$:

$$G_s(T_1, \ldots, T_N) = 0, s = 1, \ldots, m,$$

$$Z_iF_0(T_1, \ldots, T_N) - F_i(T_1, \ldots, T_N) = 0, i = 1, \ldots, n.$$

It is easy to compute the Jacobian matrix. We get

$$\begin{pmatrix}
\frac{\partial G_1}{\partial T_1}(y,z) & \ldots & \frac{\partial G_1}{\partial T_N}(y,z) & 0 & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\frac{\partial G_m}{\partial T_1}(y,z) & \ldots & \frac{\partial G_m}{\partial T_N}(y,z) & 0 & \ldots & \ldots & 0 \\
z_1\frac{\partial F_0}{\partial T_1}(y) - \frac{\partial F_1}{\partial T_1}(y) & \ldots & z_1\frac{\partial F_0}{\partial T_N}(y) - \frac{\partial F_1}{\partial T_N}(y) & -\frac{\partial F_0}{\partial T_1}(y) & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
z_1\frac{\partial F_0}{\partial T_1}(y) - \frac{\partial F_n}{\partial T_1}(y) & \ldots & z_1\frac{\partial F_0}{\partial T_N}(y) - \frac{\partial F_n}{\partial T_N}(y) & -\frac{\partial F_n}{\partial T_N}(y) & 0 & \ldots & 0
\end{pmatrix}$$

We see that the submatrix J_1 of J formed by the first N columns is obtained from the Jacobian matrix of Y computed at the point y by applying elementary row transformations and when deleting the row corresponding to the polynomial F_0. Since Y is nonsingular at y, the rank of J_1 is greater or equal than $N - \dim_x Y - 1 = N - \dim X + n$. So rank $J \geq N + n - \dim X = N + n - \dim B_Y(X)$. This implies that $B_Y(X)$ is nonsingular at the point (y,z).

Remarks. 2. The pre-image $E = \sigma^{-1}(Y)$ of Y is called the *exceptional divisor* of the blowing up $\sigma : B_Y(X) \to X$. The map σ "blows down" E of $B_Y(X)$ to the closed subset Y of X of codimension $n+1$.

3. Lemma 2 allows us to "globalize" the definition of the blow-up. Let X be any quasi-projective algebraic set and Y be its closed subset. For every affine open set $U \subset X, Y \cap U$ is a closed subset of U and the blow-up $B_Y \cap U(U)$ is defined. It can be shown that for any open affine cover $\{U_i\}_{i \in I}$ of X, the blowing-ups $\sigma_i : B_{U_i \cap Y}(U_i) \to U_i$ and $\sigma_j : B_{U_j \cap Y}(U_j) \to U_j$ can be "glued together" along their isomorphic open subsets $\sigma_i^{-1}(U_i \cap U_j) \cong \sigma_j^{-1}(U_j \cap U_j)$. Using more techniques one can show that there exists a quasi-projective algebraic set $B_Y(X)$ and a regular map $\sigma : B_Y(X) \to X$ such that $\sigma^{-1}(U_i) \cong B_{U_i \cap Y}(U_i)$ and, under this isomorphism, the restriction of σ to $\sigma^{-1}(U_i)$ coincides with σ_i.

The next fundamental results about blow-ups are stated without proof.

Theorem 2. *Let $f : X - \to Y$ be a rational map between two quasi-projective algebraic sets. There exists a closed subset Z of X and a regular map $f' : B_Z(X) \to Y$ such that f' is equal to the composition of the rational map $\sigma : B_Z(X) \to X$ and f.*

Although it sounds nice, the theorem gives very little. The structure of the blowing-up along an arbitrary closed subset is very complicated and hence this theorem gives little insight into the structure of any birational map. It is conjectured that every birational map between two nonsingular algebraic sets is the composition of blow-ups along nonsingular subsets and of their inverses. It is known for surfaces and, under some restriction, for threefolds.

Definition. A birational regular map $\sigma : \bar{X} \to X$ of algebraic sets is said to be a *resolution of singularities* of X if $\bar{X}$ is nonsingular and σ is an isomorphism over any open set of X consisting of nonsingular points.

The next fundamental result of Heisuki Hironaka brought him the Fields Medal in 1966:

Theorem 3. *Let X be an irreducible algebraic set over an algebraically closed field k of characteristic 0. There exists a sequence of monoidal transformations $\sigma_i : X_i \to X_{i-1}, i = 1, \ldots, n$, along nonsingular closed subsets of X_{i-1} contained in the set of singular points of X_{i-1}, and such that the composition $X_n \to X_0 = X$ is a resolution of singularities.*

A most common method for define a resolution of singularities is to embed a variety into a nonsingular one, blow up the latter and see what happens with the proper inverse transform of the subvariety (embedded resolution of singularities).

Definition. Let $\sigma : X \to Y$ be a birational regular map of irreducible algebraic sets, Z be a closed subset of X. Assume that σ is an isomorphism over an open subset U of X. The proper inverse transform of Z under σ is the closure of $\sigma^{-1}(U \cap Z)$ in X.

Clearly, the restriction of σ to the proper inverse transform Z' of Z is a birational regular map and $Z' = \sigma^{-1}(Z \cap U) \cup (Z' \cap \sigma^{-1}(X \setminus U)$.

Example 3. Let $\sigma : B = B_{\{0\}}(\mathbf{A}^2(K)) \to \mathbf{A}^2(K)$ be the blowing up of the origin $0 = V(Z_1, Z_2)$ in the affine plane. Let

$$Y = V(Z_2^2 - Z_1^2(Z_1 + 1)).$$

The pre-image $\sigma^{-1}(Y)$ is the union of the proper inverse transform $\bar{\sigma}^{-1}(Y)$ of Y and the fibre $\sigma^{-1}(0) \cong \mathbf{P}^1(K)$. Let us find $\bar{\sigma}^{-1}(Y)$. Recall that B is the union of two affine pieces:

$$U = V(Z_2 - Z_1 t) \subset X \times \mathbf{P}^1(K)_0, t = T_1/T_0,$$

$$V = V(Z_2 t' - Z_1) \subset X \times \mathbf{P}^1(K)_1, t' = T_0/T_1.$$

The restriction σ_1 of σ to U is the regular map $U \to \mathbf{A}^2(K)$ given by the homomorphism of rings:

$$\sigma_1^* : k[Z_1, Z_2] \to \mathcal{O}(U) = k[Z_1, Z_2, t]/(Z_2 - Z_1 t) \cong k[Z_1, t].$$

The pre-image of Y in U is the set of zeroes of the function

$$\sigma_1^*(Z_2^2 - Z_1^2(Z_1 + 1)) = Z_1^2(t^2 - Z_1 - 1)).$$

Similarly, the restriction σ_2 of σ to V is a regular map $V \to \mathbf{A}^2(K)$ given by the homomorphism of rings:

$$\sigma_2^* : k[Z_1, Z_2] \to \mathcal{O}(U) = k[Z_1, Z_2, t]/(Z_2 t' - Z_1) \cong k[Z_2, t'].$$

The pre-image of Y in V is the set of zeroes of the function

$$\sigma_2^*(Z_2^2 - Z_1^2(Z_1 + 1)) = Z_2^2(1 - t'^2(Z_2 t' + 1)).$$

Thus

$$\sigma^{-1}(Y) \cap U = E_1 \cup C_1, \sigma^{-1}(Y) \cap V = E_2 \cup C_2,$$

where

$$E_1 = V(Z_1), C_1 = V(t^2 - Z_1 - 1) \subset U \cong \mathbf{A}^2(K),$$

$$E_2 = V(Z_2), C_2 = V(1 - t'^2(Z_2 t' + 1)) \subset V \cong \mathbf{A}^2(K).$$

It is clear that

$$E_1 = \sigma^{-1}(0) \cap U \cong \mathbf{A}^1(K), E_2 = \sigma^{-1}(0) \cap V \cong \mathbf{A}^1(K),$$

i.e.,$\sigma^{-1}(0) = E_1 \cup E_2 \cong \mathbf{P}^1(K)$. Thus the proper inverse transform of Y is equal to the union $C = C_1 \cup C_2$. By differentiating we find that both C_1 and C_2 are nonsingular curves, hence C is nonsingular. Moreover,

$$C_1 \cap \sigma^{-1}(0) = V(Z_1, t^2 - 1) = \{(0,1), (0,-1)\},$$

$$C_2 \cap \sigma^{-1}(0) = V(Z_2, t'^2 - 1) = \{(0,1), (0,-1)\}.$$

Note that since $t = t'^{-1}$ at $U \cap V$, we obtain $C_1 \cap \sigma^{-1}(0) = C_2 \cap \sigma^{-1}(0)$. Hence $\sigma^{-1}(0) \cap C$ consists of two points. Moreover, it is easy to see that the curve C intersects the exceptional divisor $E = \sigma^{-1}(0)$ transversally at the two points. So the picture is as follows:

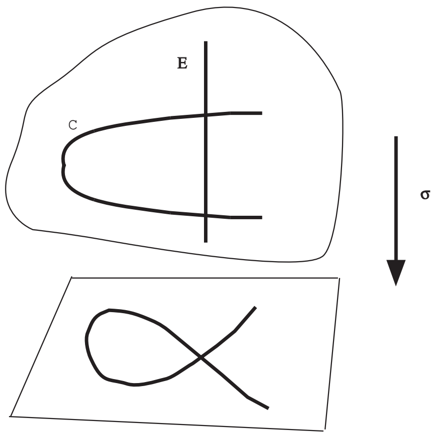

Fig.2

The restriction $\sigma : C \to Y$ is a resolution of singularities of Y.

Example 4. This time we take $Y = V(Z_1^2 - Z_2^3)$. We leave to the reader to repeat everything we have done in Example 1 to verify that the proper transform $\bar{\sigma}^{-1}(Y)$ is nonsingular and is tangent to the exceptional divisor E at one point. So, the picture is like this

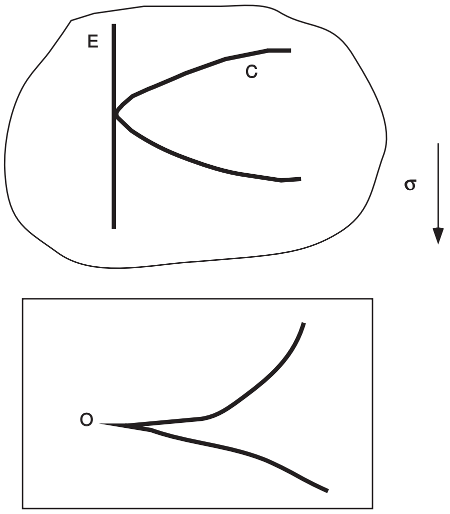

Fig.3

Example 5. Let $Y = V(F(Z_1, \dots, Z_n)) \subset \mathbf{A}^n(K)$, where F is a homogeneous polynomial of degree d. We say that Y is a cone over $\bar{Y} = V(F(Z_1, \dots, Z_n)$ in $\mathbf{P}^{n-1}(K)$. If identify $\mathbf{A}^n(K)$ with $\mathbf{P}^n(K)_0$, and $\bar{Y}$ with the closed subset $V(Z_0, F) \subset V(Z_0) \cong \mathbf{P}^{n-1}(K)$, we find that Y is the union of the lines joining the point $(1, 0, \dots, 0)$ with points in $\bar{Y}$. Let $\sigma : B = B_{\{0\}}(\mathbf{A}^n(K)) \to \mathbf{A}^n(K)$ be the blowing up of the origin in $\mathbf{A}^n(K)$. Then

$$B = \cup_i U_i, U_i = B \cap \mathbf{A}^n(K) \times \mathbf{P}^{n-1}(K)_i,$$

and

$$\sigma^{-1}(Y) \cap U_i = V(F(Z_1, \dots Z_n)) \cap V(\{Z_j - t_j Z_i\}_{j \neq i}) \cong V(Z_i^d G(t_1, \dots, t_{n-1})),$$

where $t_j = T_j/T_0$, and G is obtained from F via dehomogenization with respect to T_i. This easily implies that

$$\sigma^{-1}(Y) = \bar{\sigma}^{-1}(Y) \cup \sigma^{-1}(0), \quad \bar{\sigma}^{-1}(Y) \cap \sigma^{-1}(0) \cong \bar{Y}.$$

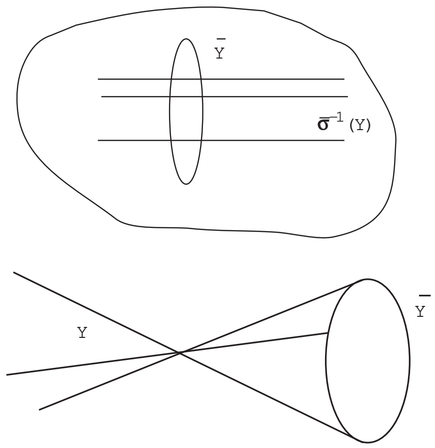

Fig.4

Example 6. Let $X = V(Z_1^2 + Z_2^3 + Z_3^4) \subset \mathbf{A}^3(K)$ and let $Y_1 = B_{\{0\}}(\mathbf{A}^3(K))$ be the blow-up. The full inverse transform of X in Y_1 is the union of three affine open subsets each isomorphic to a closed subset of $\mathbf{A}^3(K)$:

$$V_1 : Z_1^2(1 + U^3 Z_1 + V^4 Z_1^2) = 0,$$

$$V_2 : Z_2^2(U^2 + Z_2 + V^4 Z_2^2) = 0,$$

$$V_3 : Z_3^2(U^2 + V^3 Z_3 + Z_3^2) = 0.$$

The equations of the proper inverse transform X_1 are obtained by dropping the first factors. In each piece V_i the equations $Z_i = 0$ define the intersection of the proper inverse transform X_1 of X with the exceptional divisor $E_1 \cong \mathbf{P}^2(K)$. It is empty set in V_1, the affine line $U = 0$ in V_2 and V_3. The fibre of the map $X_1 \to X$ over the origin is $R_1 \cong \mathbf{P}^1(K)$. It is easy to see (by differentiation) that V_1 and V_2 are nonsingular but V_3 is singular at the point $(U, V, Z_3) = (0, 0, 0)$. Now let us start again. Replace X by $V_3 \cong V(Z_1^2 + Z_2^3 Z_3 + Z_3^2) \subset \mathbf{P}^3(K)$ and blow-up the origin. Then glue the blow-up with V_1 and V_2 along $V_3 \cap (V_1 \cup V_2)$. We obtain that the proper inverse transform X_2 of X_1 is covered by V_1, V_2 as above and three more pieces

$$V_4 : 1 + U^3 V Z_1^2 + V^3 Z_1 = 0$$

$$V_5 : U^2 + Z_2^2 V + V^2 = 0,$$

$$V_6 : U^2 + V^3 Z_3^2 + 1 = 0.$$

The fibre over the origin is the union of two curves R_2, R_3 each isomorphic to $\mathbf{P}^1(K)$. The equation of $R_2 \cup R_3$ in V_5 is $U^2 + V^2 = 0$. The equation of $R_2 \cup R_3$ in V_3 is $U^2 + 1 = 0$. Since $R_1 \cap V_3$ was given by the equation $Z_3 = 0$ and we used the substitution $Z_3 = V Z_2$ in V_5, we see that the pre-image of R_1 intersects R_1 and R_2 at their common point $(U, V, Z_2) = (0, 0, 0)$ in V_5. This point is the unique singular point of X_2. Let us blow-up the origin in V_5. We obtain X_3 which is covered by open sets isomorphic to V_1, V_2, V_4, V_6 and three more pieces:

$$V_7 : 1 + V U^2 Z_1 + V^2 = 0,$$

$$V_8 : U^2 + V^2 Z_3 + 1.$$

$$V_9 : U^2 + V Z_2 + V^2 = 0,$$

The pre-image of the origin in the proper inverse transform X_3 of X_2 consists of two curves R_4, R_5 each isomorphic to $\mathbf{P}^1(K)$. In the open set V_9 they are given by the equations $V = 0, U = \pm\sqrt{-1}V$. The inverse image of the curve R_1 intersects R_4, R_5 at their intersection point. The inverse images of R_2 intersects R_4 at the point $(U, V, Z_2) = (1, \sqrt{-1}, 0)$, the inverse image of R_3 intersects R_5 at the point $(1, -\sqrt{-1}, 0)$. Finally we blow up the origin at V_9 and obtain that the proper-inverse transform X_4 is nonsingular. It is covered by opne affine subsets isomorphic to $V_1, \ldots, V_8$ and three more open sets

$$V_{10} : 1 + UV + V^2 = 0,$$

$$V_{11} : U^2 + V + V^2 = 0,$$

$$V_{12} : U^2 + V + 1 = 0.$$

The pre-image of the origin in X_4 is a curve $R_6 \cong \mathbf{P}^1(K)$. It is given by the homogeneous equation $T_0^2 + T_1 T_2 + T_2^2$ in homogeneous coordinates of the exceptional divisor of the blow-up (compare it with Example 5). The image of the curve R_1 intersects R_6 at one point. So we get a resoluton of singularities $\sigma : \bar{X} = X_4 \to X$ with σ^{-1} equal to the union of six curves each isomorphic to projective line. They intersect each other according to the picture:

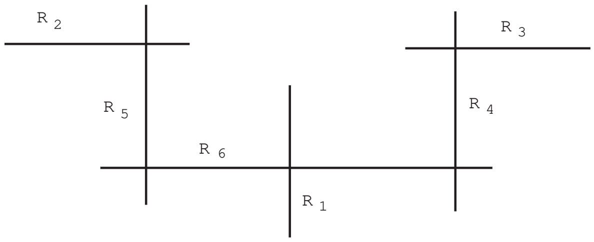

Fig.5

Let Γ be the graph whose vertices correspond to irreducible components of $\sigma^{-1}(0)$ and edges to intersection points of components. In this way we obtain the graph

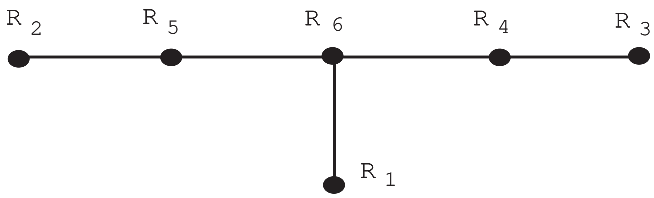

Fig.6

It is the Dynkin diagram of simple Lie algebra of type E_6.

Exercises.

1. Prove that $B_{V(I)}(X)$ is not affine unless I is (locally) a principal ideal.
2. Resolve the singularities of the curve $x^n + y^r = 0, (n, r) = 1$, by a sequence of blow-ups in the ambient space. How many blow-ups do you need to resolve the singularity?
3. Resolve the singularity of the affine surface $X : Z_1^2 + Z_2^3 + Z_3^3 = 0$ by a sequence of blow-ups in the ambient space. Describe the exceptional curve of the resolution $f : \bar{X} \to X$.
4. Describe $A(I)$, where $A = k[Z_1, Z_2,], I = (Z_1, Z_2^2)$. Find the closed subset $B_I(A)$ of $\mathbf{A}^2(K) \times \mathbf{P}^1(K)$ defined by the kernel of the homomorphism $\phi : A[T_0, T_1] \to A(I), T_0 \to Z_1, T_2 \to Z_2^2$. Is it nonsingular?

5*. Resolve the singularities of the affine surface $X : Z_1^2 + Z_2^3 + Z_3^5 = 0$ by a sequence of blow-ups in the ambient space. Show that one can find a resolution of singularities $f : \bar{X} \to X$ such that the graph of irreducible components of $f^{-1}(0)$ is the Dynkin diagram of the root system of a simple Lie algebra of type E_8.

6*. Resolve the singularities of the affine surface $X : Z_1 Z_2^3 + Z_1^3 + Z_3^2 = 0$ by a sequence of blow-ups in the ambient space. Show that one can find a resolution of singularities $f : \bar{X} \to X$ such that the graph of irreducible components of $f^{-1}(0)$ is the Dynkin diagram of the root system of a simple Lie algebra of type E_7.

7*. Resolve the singularities of the affine surface $X : Z_1(Z_2^2 + Z_1^n) + Z_3^2 = 0$ by a sequence of blow-ups in the ambient space. Show that one can find a resolution of singularities $f : \bar{X} \to X$ such that the graph of irreducible components of $f^{-1}(0)$ is the Dynkin diagram of the root system of a simple Lie algebra of type D_n.

8*. Resolve the singularities of the affine surface $X : Z_1Z_2^2 + Z_3^{n+1} = 0$ by a sequence of blow-ups in the ambient space. Show that one can find a resolution of singularities $f : \bar{X} \to X$ such that the graph of irreducible components of $f^{-1}(0)$ is the Dynkin diagram of the root system of a simple Lie algebra of type A_n.

9*. Let $f : \mathbf{P}^2(K) - \to \mathbf{P}^2(K)$ be the rational map given by the formula $T_0 \to T_1T_2, T_1 \to T_2T_3, T_2 \to T_0T_1$. Show that there exist two birational regular maps $\sigma_1, \sigma_2 : X \to \mathbf{P}^2(K)$ with $f \circ \sigma_1 = \sigma_2$ such that the restriction of each σ_i over $\mathbf{P}^2(K)_j, j = 0, 1, 2$ is isomorphic to the blow-up along one point.